

# A SIMPLICIAL CALCULUS FOR LOCAL INTERSECTION NUMBERS AT NONARCHIMEDIAN PLACES ON PRODUCTS OF SEMI-STABLE CURVES

JOHANNES KOLB

**ABSTRACT.** We analyse the subring of the Chow ring with support generated by the irreducible components of the special fibre of the Gross-Schoen desingularization of a  $d$ -fold self product of a semi-stable curve over the spectrum of a discrete valuation ring. For this purpose we develop a calculus which allows to determine intersection numbers in the special fibre explicitly. As input our simplicial calculus needs only combinatorial data of the special fibre. It yields a practical procedure for calculating even self intersections in the special fibre. The first ingredient of our simplicial calculus is a localization formula, which reduces the problem of calculating intersection numbers to a special situation. In order to illustrate how our simplicial calculus works, we calculate all intersection numbers between divisors with support in the special fibre in dimension three and four. The localization formula and the general idea were already presented for  $d = 2$  in a paper of Zhang [Zha10, Ch. 3]. In our present work we achieve a generalisation to arbitrary  $d$ .

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## 1. INTRODUCTION

Let  $R$  be a complete discrete valuation ring with algebraically closed residue class field  $k$ . We denote the quotient field  $\text{Quot}(R)$  by  $K$  and a uniformizing element with  $\pi \in R$ . Furthermore let  $S$  denote the scheme  $\text{Spec } R$  with generic point  $\eta$  and special point  $s$ . Let  $X$  be a regular strict semi-stable  $S$ -scheme. We denote by

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*Date:* April 8, 2014.

$\text{CaDiv}_{X_s}(W)$  the group of Cartier divisors on  $X$  with support in the special fibre  $X_s$ . Intersection theory with support yields a product

$$\left(\text{CaDiv}_{X_s}(X)\right)^p \rightarrow \text{CH}^{p-1}(X_s),$$

where  $\text{CH}^p$  denotes the Chow group in codimension  $p$ . If  $X/S$  is proper of dimension  $\dim(X) = d + 1$ , then  $X_s$  is proper over a field and therefore there is a degree map  $\text{ldeg} : \text{CH}^d(X_s) \rightarrow \mathbb{Z}$ . We are interested in the pairing

$$(1.1) \quad \begin{aligned} &(\text{CaDiv}_{X_s}(X))^{d+1} \rightarrow \mathbb{Z}, \\ &(C_0, \dots, C_d) \mapsto \text{ldeg}(C_0 \cdots C_d) \end{aligned}$$

given by the intersection pairing and the degree map.

Let us first look at a simple example: Assume that  $X$  is a regular strict semi-stable model of a smooth proper curve  $X_\eta$  over  $K$ . By the semi-stable reduction theorem, each smooth curve over  $K$  has a regular strict semi-stable model after finite base-change. We denote by  $\Gamma(X)$  the dual graph of  $X_s$ . Then the pairing (1.1) can be calculated by counting suitable edges in  $\Gamma(X)$ : For instance, the self-intersection number of an irreducible component  $C \subseteq X_s$  is given by the number of edges in  $\Gamma(X)$  connected to  $C$ .

We deduce an analog to this description in the following higher-dimensional setting, in which an explicit construction of a semi-stable model is still possible: Let  $X$  be a regular strict semi-stable model of a smooth curve over  $K$ . Then a model of  $(X_\eta)^d$  is given by the  $d$ -fold product  $X \times_R \cdots \times_R X$ . As already shown by Gross and Schoen ([Gro95]), this model can be desingularized to a regular strict semi-stable scheme. Using as additional data an ordering on the set  $X_s^{(0)}$  of irreducible components of  $X_s$  we can make this desingularization canonical, therefore we get a well-defined desingularization  $W$  of the scheme  $X^d$ .

As replacement for the reduction graph in higher dimension we use the incidence relations in  $W_s$  to define a simplicial set  $\mathcal{R}(W)$ , the simplicial reduction set. The underlying set of its geometric realization is just the product  $\mathcal{R}(W) = \Gamma(X)^d$  endowed with a triangulation depending on the chosen order on  $X_s^{(0)}$ . We introduce a calculus on the subring of the Chow group with support in the special fibre, which is generated by the classes of the irreducible components of  $W_s$ , which enables us to calculate intersection numbers of components of the special fibre. This simplicial calculus uses only explicitly given rational equivalences which arise from the combinatoric of  $\mathcal{R}(W)$ . It is therefore described using a ring  $\mathcal{C}(\mathcal{R}(W))$ , which is generated by the 0-simplices in  $\mathcal{R}(W)$  (see definition 4.12).

The calculus determines intersection numbers in a very controlled way, which ensures that the numbers are localized in a certain way. It is enough to determine these numbers in the following local situation: Let  $\tilde{L}$  denote the regular strict semi-stable model  $\tilde{L} := \text{Proj}(x_0x_1 - \pi z^2)$ . The special fibre  $\tilde{L}_s$  consists of two irreducible components  $\text{div}(x_0)$  and  $\text{div}(x_1)$ , which we order by  $\text{div}(x_0) < \text{div}(x_1)$ . Let  $d \in \mathbb{N}$  and  $M$  be the Gross-Schoen desingularization described above of the product  $L^d$ . Then  $\mathcal{R}(M)$  is a cube with its the standard desingularization.

The reduction simplicial set  $\mathcal{R}(W)$  can be split into such cubes: Denote by  $\Gamma(X)^1$  the edges of the graph  $\Gamma(X)$ . For each tuple  $\gamma = (\gamma_1, \dots, \gamma_d)$  with  $\gamma_1, \dots, \gamma_d \in \Gamma(X)^1$  there exists by functoriality an associated embedding  $i_\gamma : \mathcal{R}(M) \rightarrow \mathcal{R}(W)$ , which induces a morphism  $i_\gamma^* : \mathcal{C}(\mathcal{R}(W)) \rightarrow \mathcal{C}(\mathcal{R}(M))$ . These morphisms localise the problem of calculating intersection numbers:

**Theorem 1.1.** *Let  $\alpha \in \mathcal{C}(W)^{d+1}$ . Then the equation*

$$\text{ldeg}_W(\alpha) = \sum_{\gamma=(\gamma_1, \dots, \gamma_d) \in (\Gamma(X)^1)^d} \text{ldeg}_M(i_\gamma^*(\alpha))$$

holds.

This theorem justifies a closer look on the local situation  $M$ . We may subscript the vertices of  $\mathcal{R}(M)$  with coordinate vectors  $v \in \mathbb{F}_2^d$ . Thus  $\{C_v \mid v \in \mathbb{F}_2^d\}$  is a basis of  $\mathcal{C}(\mathcal{R}(M))^1$ . The discrete Fourier transforms

$$F_v := \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v, w \rangle} C_w$$

yield another basis of  $\mathcal{C}(\mathcal{R}(M))_{\mathbb{Q}}^1 := \mathcal{C}(\mathcal{R}(M))^1 \otimes_{\mathbb{Z}} \mathbb{Q}$ . It turns out that in this basis the intersection numbers are relatively easy to describe for  $d \in \{2, 3\}$ :

**Theorem 4.32.** *Let  $d = 2$  and  $v_1, v_2, v_3$  vectors in  $\mathbb{F}_2^2$ . Then the following holds:*

$$\text{ldeg}(F_{v_1} F_{v_2} F_{v_3}) = \begin{cases} -32 & \text{if } v_1 = v_2 = v_3 = (1, 1), \\ 16 & \text{if } \{v_1, v_2, v_3\} = \{(1, 0), (0, 1), (1, 1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Using a different desingularization procedure, Zhang already studied a similar situation in [Zha10, 3.1].

Note that for the case  $d = 3$  the symmetric group  $S_3$  acts on vectors in  $\mathbb{F}_2^3$  by permuting the standard basis and the symmetric group  $S_4$  acts on tuples  $(v_0, \dots, v_3)$  by a permutation of the elements. For  $\sigma \in S_4, \tau \in S_3$  denote the successive application of both operations by

$$(v_0, \dots, v_3)^{\sigma, \tau} := (v_{\sigma(0)}^{\tau}, \dots, v_{\sigma(3)}^{\tau}).$$

**Theorem 4.33.** *Let  $V = (v_0, \dots, v_3) \in (\mathbb{F}_2^3)^4$  be a 4-tuple of vectors in  $\mathbb{F}_2^3$ . Then the intersection numbers in  $\mathcal{C}(I^3)_{\mathbb{Q}}$  are*

$$\text{ldeg}(F_{v_0} F_{v_1} F_{v_2} F_{v_3}) = \begin{cases} -64 & \text{if } V = (100, 010, 001, 111)^{\sigma, \tau} \text{ for } \sigma \in S_4, \tau \in S_3, \\ -64 & \text{if } V = (100, 010, 101, 011)^{\sigma, \tau} \text{ for } \sigma \in S_4, \tau \in S_3, \\ -64 & \text{if } V = (100, 110, 101, 111)^{\sigma, \tau} \text{ for } \sigma \in S_4, \tau \in S_3, \\ 128 & \text{if } V = (100, 011, 011, 111)^{\sigma, \tau} \text{ for } \sigma \in S_4, \tau \in S_3, \\ 128 & \text{if } V = (100, 111, 111, 111)^{\sigma, \tau} \text{ for } \sigma \in S_4, \tau \in S_3, \\ 128 & \text{if } V = (110, 110, 101, 011)^{\sigma, \tau} \text{ for } \sigma \in S_4, \tau \in S_3, \\ -128 & \text{if } V = (110, 101, 111, 111)^{\sigma, \tau} \text{ for } \sigma \in S_4, \tau \in S_3, \\ 512 & \text{if } V = (111, 111, 111, 111)^{\sigma, \tau} \text{ for } \sigma \in S_4, \tau \in S_3, \\ 0 & \text{otherwise.} \end{cases}$$

Especially interesting is the fact that many of the intersection numbers vanish. We may therefore propose a vanishing conjecture, which we will describe in the following:

**Definition 1.2.** Let  $\mathcal{P} = \{P_1, \dots, P_l\}$  be a partition of the set  $\{1, \dots, d\}$  and  $v = (v_1, \dots, v_d) \in \mathbb{F}_2^d$ . Then set

$$\alpha(\mathcal{P}, v) := \#\{i \in \{1, \dots, l\} \mid \exists j \in P_i, v_j = 1\}.$$

**Definition 4.35.** Let  $d \in \mathbb{N}$ . We say that  $d$  verifies the vanishing condition, iff for each  $\mathcal{P}$  a partition of  $\{1, \dots, d\}$  and  $v_0, \dots, v_d \in \mathbb{F}_2^d$  with

$$\sum_i \alpha(\mathcal{P}, v_i) < d + |\mathcal{P}|$$

the intersection number

$$\text{ldeg}\left(\prod_i F_{v_i}\right)$$

vanishes.

**Vanishing Conjecture.** *The vanishing condition holds for arbitrary  $d \in \mathbb{N}$ .*

By [theorem 4.32](#) and [theorem 4.33](#) this conjecture is true for  $d = 2$  and  $d = 3$ . Using the computer algebra system Sage [\[Ste13\]](#) we were also able to verify the cases  $d = 4$  and  $d = 5$ . This computer verification is a first demonstration of the power of the simplicial calculus developed in this paper.

Each verification of the vanishing conjecture for a particular  $d$  has non-trivial consequences which were our original motivation to consider this conjecture. Namely we can derive a formula for the arithmetic intersection numbers of suitable adelic metrized line bundles on the  $d$ -fold self-product of any curve in purely combinatorial and elementary analytic terms on the associated reduction complex. Details can be found in [\[Kol\]](#).

## 2. REGULAR STRICT SEMI-STABLE SCHEMES

For regular strict semi-stable curves the reduction graph is a good combinatorial invariant. We generalise this concept for arbitrary regular strict semi-stable schemes and define the simplicial reduction set. This combinatorial object is used later to compute intersection numbers in the special fibre.

Let  $R$  denote a complete discrete valuation ring with algebraically closed residue field  $k$ . The scheme  $S := \operatorname{Spec} R$  consists of the generic point  $\eta$  and the special point  $s$ . Furthermore let  $\pi$  denote a uniformizer of  $R$ .

Let  $X$  be a scheme. The set of points of codimension  $q$  is denoted by  $X^{(q)}$ . In particular, we get the irreducible components by  $X^{(0)} := \{p \in X \mid \dim \mathcal{O}_{X,p} = 0\}$ .

**Definition 2.1.** [\[dJ96, 2.16\]](#) Let  $S := \operatorname{Spec} R$  be the spectrum of a discrete valuation ring  $R$  with algebraic closed residue field  $k$ . Let  $X$  be an integral flat and separated  $S$ -scheme of finite type. We call  $X$  *regular strict semi-stable*, if the following properties hold:

- (i) The generic fibre  $X_\eta$  is smooth,
- (ii) the special fibre  $X_s$  is reduced,
- (iii) each irreducible component  $C$  of  $X_s$  is a Cartier divisor on  $X$ , and
- (iv) if  $C_1, \dots, C_m$  is a subset of irreducible components of  $X_s$ , then the scheme-theoretic intersection  $C_1 \cap \dots \cap C_m$  is either smooth or empty.

A regular strict semi-stable  $S$ -variety of dimension 2 has relative dimension 1 over  $S$  and is therefore called *regular strict semi-stable curve over  $S$* .

**Example 2.2.** The affine scheme  $\operatorname{Spec} R[x_0, \dots, x_n]/(x_0 \cdots x_n - \pi)$  is a regular strict semi-stable  $S$ -variety.

According to Urs Hartl [\[Har01\]](#) every regular strict semi-stable  $S$ -variety is of this type:

**Theorem 2.3.** *A  $S$ -variety  $X$  is regular strict semi-stable iff the generic fibre  $X_\eta$  is smooth and for each closed point  $x \in X_s$  exists an open neighbourhood  $U$  in  $X$ , a number  $m \in \mathbb{N}$  and a smooth morphism*

$$f : U \rightarrow L_m := \operatorname{Spec} R[x_0, \dots, x_m]/(x_0 \cdots x_m - \pi).$$

*The morphism can be chosen such that it maps the point  $x$  to the “origin”, i.e., the point  $p_0$  given by the ideal  $(x_0, \dots, x_m)$ .*

*Proof.* [\[Har01, Prop 1.3\]](#) □

We use this fact in form of the following easy corollaries:

**Corollary 2.4.** *Let  $X, Y$  be integral, flat, separated  $S$ -schemes of finite type and  $f : X \rightarrow Y$  a smooth morphism. If  $Y$  is regular strict semi-stable, then so is  $X$ .*

**Corollary 2.5.** *Let  $X$  be a regular strict semi-stable  $S$ -curve. Then each closed point  $x \in X_s$  is either smooth or has an open neighbourhood  $U$  and an étale map*

$$U \rightarrow \operatorname{Spec} R[x_0, x_1]/(x_0x_1 - \pi).$$

*Proof.* According to [theorem 2.3](#) there is a smooth morphism

$$f : U \rightarrow \operatorname{Spec} R[x_0, \dots, x_m]/(x_0 \cdots x_m - \pi)$$

of an open neighbourhood  $U$  of  $x$ . By dimension theory only  $m = 0$  and  $m = 1$  is possible. If  $m = 0$ , then  $U \rightarrow S$  is smooth. If otherwise  $m = 1$ , the smooth morphism  $f$  has relative dimension 0 and is therefore étale.  $\square$

It follows from [corollary 2.5](#) that for étale local questions we can restrict ourselves to the model scheme  $L = L_1 := \operatorname{Spec} R[x_0, x_1]/(x_0x_1 - \pi)$ . The special fibre of this scheme consists of two components, which have a proper intersection in one point, the “origin” given by the ideal  $(x_0, x_1)$ .

Let  $X$  be a regular strict semi-stable scheme. A set of pairwise different components of the special fibre  $X_s$  intersects properly by [definition 2.1\(iv\)](#) and we will show in [proposition 4.8](#) that this is an intersection of multiplicity 1 (in the sense of intersection theory). We may thus expect that a part of the Chow group is determined only by the incidence relations between the components. To get better functorial properties we endow these incidence relations with the structure of a simplicial set, the simplicial reduction set.

For this definition it is necessary to choose a total ordering on  $X_s^{(0)}$ , the components of the special fibre. This is a transitive, antisymmetric and reflexive relation  $\leq$ , by which each two elements  $C_1, C_2 \in X_s^{(0)}$  are comparable. Furthermore we employ the usual definitions from the theory of simplicial sets: By  $\Delta$  we denote the simplicial category, this is the category consisting of the ordered sets  $[n] := \{0, \dots, n\}$  for each  $n \in \mathbb{N}_0$  as objects and monotonically increasing maps as morphisms. Some basic facts about partial orders and the simplicial category are outlined in [appendix A](#).

**Definition 2.6.** Let  $X$  be a regular strict semi-stable  $S$ -scheme and  $\leq$  a total ordering on  $X_s^{(0)}$ . For each morphism of ordered sets  $\beta : [n] \rightarrow X_s^{(0)}$ , i.e., a monotonically increasing map, we denote the scheme-theoretic intersection

$$[\beta] := \beta(0) \cap \cdots \cap \beta(n)$$

by  $[\beta]$ .

*Remark 2.7.* Let  $X, \leq$  be as above,  $\beta : [m] \rightarrow X_s^{(0)}$  a morphism of ordered sets and  $f : [n] \rightarrow [m]$  a morphism of the simplicial category  $\Delta$ . Then

$$[\beta] \subseteq [\beta \circ f]$$

holds. Since  $[\beta]$  is smooth over  $k$  ([definition 2.1\(iv\)](#)), the irreducible components  $[\beta]^{(0)}$  are actually connected components ([\[Liu02, Cor 4.2.17\]](#)) and therefore there is a canonical morphism

$$(2.1) \quad f_\beta : [\beta]^{(0)} \rightarrow [\beta \circ f]^{(0)}$$

which maps each point from  $[\beta]^{(0)}$  onto its containing connected component from  $[\beta \circ f]$ .

**Definition 2.8.** Let  $X$  be a regular strict semi-stable scheme on  $S$  and  $\leq$  a total ordering on  $X_s^{(0)}$ . The *simplicial reduction set* of  $X$  is the simplicial set  $\mathcal{R}(X) : \Delta \rightarrow \operatorname{Set}$  defined on objects  $[n] \in \Delta$  by

$$\mathcal{R}(X)_n := \mathcal{R}(X)([n]) := \coprod_{\beta \in \operatorname{hom}([n], X_s^{(0)})} [\beta]^{(0)}$$

and on morphisms  $f : [n] \rightarrow [m]$  by

$$\mathcal{R}(X)(f) = \left[ \coprod_{\beta \in \text{hom}([m], X_s^{(0)})} f_\beta \right] : \mathcal{R}(X)_m \rightarrow \mathcal{R}(X)_n.$$

In the last equation  $f_\beta$  is the map from (2.1).

If  $\dim(X) = 2$ , i.e.,  $X$  is a  $S$ -curve, we call  $\mathcal{R}(X)$  also *reduction graph* and denote it by  $\Gamma(X)$ .

A simplicial set is determined easily if each simplex is uniquely given by its vertices. We call these *simplicial sets without multiple simplices* (compare definition A.9). We may test this property using the following criterium:

**Proposition 2.9.** *Let  $X$  be a regular strict semi-stable  $S$ -scheme with a total ordering  $\leq$  on  $X_s^{(0)}$ . The simplicial set  $\mathcal{R}(X)$  is a simplicial set without multiple simplices iff for each set  $\{C_1, \dots, C_k\}$  of components of  $X_s$  the intersection  $C_1 \cap \dots \cap C_k$  is connected.*

*In this case there is a canonical bijection*

$$(2.2) \quad \mathcal{R}(X)_k \simeq \{C_0 \leq \dots \leq C_k \mid C_0, \dots, C_k \in \mathcal{R}(X)_0, C_0 \cap \dots \cap C_d \neq \emptyset\}$$

*between the  $k$ -simplices and ascending chains of components in  $(X_s)^{(0)}$  with non-empty intersection.*

*Proof.* Let  $\sigma \in \mathcal{R}(X)_n$  be an  $n$ -simplex of the reduction complex. It is given by a pair  $(\beta, p)$  with  $\beta : [n] \rightarrow X_s^{(0)}$  and  $p \in [\beta]^{(0)}$ . An easy computation shows that the vertices of  $\sigma$  are given by  $\beta(0), \dots, \beta(n)$ . The simplex  $\sigma$  is therefore uniquely determined by its vertices iff  $[\beta]^{(0)}$  is a singleton, which means  $[\beta]$  is connected. The bijection in (2.2) is then given by

$$\sigma = (\beta, p) \mapsto (\beta(0) \leq \dots \leq \beta(n)).$$

□

**Remark 2.10.** From now on we restrict ourselves to regular strict semi-stable schemes having a reduction set without multiple simplices, since proposition 2.9 gives a comfortable description of the reduction set. At least for curves this restriction is not essential: By a suitable base change  $S_n \rightarrow S$  and a subsequent desingularization each regular strict semi-stable  $S$ -curve can be transformed into a regular strict-semi-stable  $S_n$ -curve without multiple simplices. This process is recalled in section 3.

**Example 2.11.** The affine  $S$ -scheme  $L_m := \text{Spec } R[x_0, \dots, x_m]/(x_0 \cdots x_n - \pi)$  from example 2.2 is regular strict semi-stable having a simplicial reduction set without multiple simplices. The components of the special fibre of  $L_m$  are given by the ideals  $(x_i)$  ( $i=0, \dots, m$ ). We endow them with the order  $(x_i) \leq (x_j)$  where  $(i \leq j)$ . Since every intersection of such components is connected and non-empty, the simplicial reduction set  $\mathcal{R}(L_m)$  is free of multiple simplices and  $\mathcal{R}(L_m)_k = \text{Hom}_\Delta([k], [m])$ . Therefore  $\mathcal{R}(L_m)$  is the standard- $m$ -simplex  $\Delta[m]$ .

**Example 2.12.** Let  $X$  be a regular strict semi-stable  $S$ -curve. The simplicial reduction set  $\Gamma(X)$  has dimension 1 and is therefore an ordered graph. It is easy to see that  $\Gamma(X)$  coincides with the usual definition of the reduction graph [Bos90, 9.2].

The simplicial reduction set is functorial for generic flat morphisms:

**Proposition 2.13.** *Let  $X$  and  $Y$  be regular strict semi-stable  $S$ -schemes with total orderings  $\leq_X$  resp.  $\leq_Y$  on  $X_s^{(0)}$  resp.  $Y_s^{(0)}$ . Let  $f : X \rightarrow Y$  be a morphism which is flat in a open subset which contains all generic points of  $Y_s$ .*

(i) Then there is a morphism

$$(2.3) \quad \begin{aligned} f_* : X_s^{(0)} &\rightarrow Y_s^{(0)}, \\ C &\mapsto \overline{f(C)}. \end{aligned}$$

(ii) If  $\mathcal{R}(X)$  and  $\mathcal{R}(Y)$  are without multiple simplices and  $f_*$  from (2.3) preserves the order, we may extend  $f_*$  to a morphism of simplicial sets by

$$(2.4) \quad \begin{aligned} f_* : \mathcal{R}(X) &\rightarrow \mathcal{R}(Y), \\ (C_0 \leq_X \cdots \leq_X C_k) &\mapsto (\overline{f(C_0)} \leq_Y \cdots \leq_Y \overline{f(C_k)}). \end{aligned}$$

In this definition the simplices of  $\mathcal{R}(X)$  and  $\mathcal{R}(Y)$  are given by the bijection (2.2) in proposition 2.9.

*Proof.* Let  $C \in X_s^{(0)}$  be a component of  $X_s$ . For the first claim it suffices to show  $\overline{f(C)} \in Y_s^{(0)}$ . Let  $p \in X_s$  be the generic point of  $C$ . After restricting to a neighbourhood of  $p$  we may assume that  $f$  is flat. Then also the base change  $f' : X_s \rightarrow Y_s$  is flat and by [Gro65, IV, §2, Cor (2.3.5) (ii)] the closure  $\{f(p)\}$  is an irreducible component of  $Y_s$ .

For the second claim we have to show that (2.4) is well-defined. Let  $C_0 \leq_X \cdots \leq_X C_k$  be an ascending chain representing a  $k$ -simplex of  $\mathcal{R}(X)$ , i.e., with  $C_0 \cap \cdots \cap C_k \neq \emptyset$ . By assumption,  $\overline{f(C_0)} \leq_Y \cdots \leq_Y \overline{f(C_k)}$  also holds. Eventually each point of  $C_0 \cap \cdots \cap C_k$  is mapped by  $f$  onto a point of  $\overline{f(C_0)} \cap \cdots \cap \overline{f(C_k)}$ , therefore the chain  $(\overline{f(C_0)} \leq_Y \cdots \leq_Y \overline{f(C_k)})$  represents an  $k$ -simplex of  $\mathcal{R}(Y)$ .  $\square$

We may determine the simplicial reduction set locally:

**Proposition 2.14.** *Let  $X$  be a regular strict semi-stable  $S$ -scheme with total ordering  $\leq$  on  $X_s^{(0)}$ .*

- (i) *If  $U \subseteq X$  is an open subset, then  $\leq_X$  induces a total ordering  $\leq_U$  on  $U_s^{(0)}$  and there is a canonical monomorphism  $\mathcal{R}(U) \rightarrow \mathcal{R}(X)$ .*
- (ii) *Let  $\mathcal{U} = (U_i)_{i \in I}$  be a covering system of open sets for  $X$ . If for each two sets  $U_i, U_j \in \mathcal{U}$  the intersection  $U_i \cap U_j$  has a covering with sets from  $\mathcal{U}$ , then*

$$\mathcal{R}(X) = \operatorname{colim}_{i \in I} \mathcal{R}(U_i).$$

*Proof.* Claim (i) is an immediate consequence of the definition of  $\mathcal{R}(X)$ . We now show (ii). The universal property of the colimit yields a unique morphism

$$(2.5) \quad \varphi : \operatorname{colim}_{i \in I} \mathcal{R}(U_i) \rightarrow \mathcal{R}(X),$$

induced by the inclusions  $U_i \rightarrow X$ . We have to show that this is an isomorphism.

Let  $k \in \mathbb{N}$  and  $\sigma \in \mathcal{R}(X)_k$  be a  $k$ -simplex. By definition  $\sigma$  is given by  $\beta \in \operatorname{Hom}([n], X_s^{(0)})$  and  $p \in [\beta]^{(0)}$ . Since  $\mathcal{U}$  is a covering system for  $X$ , there is an  $U \in \mathcal{U}$  with  $p \in U$  and thus  $\sigma \in \mathcal{R}(U)$ . Morphism (2.5) is therefore surjective.

To proof injectivity let  $k \in \mathbb{N}$ ,  $U, U' \in \mathcal{U}$  and  $\sigma \in \mathcal{R}(U)_k$ ,  $\sigma' \in \mathcal{R}(U')_k$  be such that  $\varphi(\sigma) = \varphi(\sigma')$ . As before let  $\sigma$  be given by a pair  $(\beta, p)$  with  $\beta \in \operatorname{Hom}([n], U_s^{(0)})$ ,  $p \in [\beta]^{(0)}$  and  $\sigma'$  be given by  $(\beta', p')$ . Then  $\varphi(\sigma) = \varphi(\sigma')$  means  $p = p'$  as points of  $X$ . Thus we may choose a neighbourhood  $U'' \in \mathcal{U}$  of  $p$  with  $U'' \subseteq U \cap U'$ . The point  $p$  also defines a simplex  $\sigma'' \in \mathcal{R}(U'')$ . For the inclusions  $i : U'' \rightarrow U$  and  $i' : U'' \rightarrow U'$  we have

$$i_*(\sigma'') = \sigma, \quad i'_*(\sigma'') = \sigma'$$

and therefore  $\sigma$  and  $\sigma'$  agree in  $\operatorname{colim}_{i \in I} \mathcal{R}(U_i)$ .  $\square$



**Definition 2.15.** Let  $X$  be a regular strict semi-stable scheme and  $p \in X$  a closed point. Then an open subset  $U \subseteq X$  is called *standard neighbourhood* of  $p$ , if either  $U \subseteq X_\eta$  or there is a number  $m \in \mathbb{N}$  and a smooth morphism

$$f : U \rightarrow L_m := \operatorname{Spec} R[x_0, \dots, x_m] / (x_0 \cdots x_m - \pi)$$

such that  $f_* : \mathcal{R}(U) \rightarrow \mathcal{R}(L_m)$  from [proposition 2.13\(ii\)](#) is a bijection and  $f(p) = p_0 := (x_0, \dots, x_m, \pi)$  holds.

**Proposition 2.16.** *Let  $X$  be a regular strict semi-stable scheme,  $U \subseteq X$  an open subset and  $p \in X_s \cap U$  a closed point in the special fibre. Then there exists a standard neighbourhood of  $p$  contained in  $U$ .*

*Proof.* By [theorem 2.3](#) there is an open subset  $U'$  of  $p$  and a smooth morphism

$$f : U' \rightarrow L_m$$

with  $f(p) = p_0$ . We may shrink  $U'$  such that each component of  $U'_s$  contains  $p$  and  $U' \subseteq U$  holds. By a further restriction we may assume that for each choice of irreducible components  $C_0, \dots, C_k \in (U')_s^{(0)}$  the intersection  $C_0 \cap \cdots \cap C_k$  is connected. Thus the simplicial reduction set  $\mathcal{R}(U')$  is isomorphic to the standard- $n$ -simplex and has no multiple simplices. By [proposition A.10](#) it suffices to show that  $f$  induces a bijection  $f_* : U_s^{(0)} \rightarrow (L_m)_s^{(0)}$  on the 0-simplices:

Since  $f$  is flat, the morphism  $f_*$  exists by [proposition 2.13](#). To give the inverse map, let  $C \in (L_m)_s^{(0)}$  be an irreducible component of  $(L_m)_s$ . By assumption  $C$  is smooth and contains  $p_0$ . Therefore the scheme-theoretical preimage  $f^{-1}(C)$  is a non-empty smooth  $k$ -scheme. This means, every irreducible component of  $f^{-1}(C)$  is a connected component of  $f^{-1}(C)$ . Since  $f$  is flat, each of these components is a component  $U'_s$  and therefore contains the point  $p$ . Thus  $f^{-1}(C)$  consists of exactly one connected component of  $U'_s$  and the morphism  $f_*$  is therefore bijective.  $\square$

**Corollary 2.17.** *Let  $X$  be a regular strict semi-stable scheme. Then every open subset  $U \subseteq X$  can be covered by a system  $(U_i)_{i \in I}$  of standard neighbourhoods. Therefore we may apply [proposition 2.14 \(ii\)](#) on the covering*

$$\mathcal{U} := \{U \subseteq X \mid U \text{ is standard neighborhood of a closed point } p \in X\}$$

and therefore

$$\mathcal{R}(X) = \operatorname{colim}_{U \in \mathcal{U}} \mathcal{R}(U)$$

holds.

*Proof.* Let  $U \subseteq X$  be an open subset. We have to show that  $U$  can be covered by standard neighbourhoods. Denote by  $I$  the set of closed points of  $U_s$ . For each point  $p \in I$  we choose a standard neighbourhood  $U_p \subseteq U$  by [proposition 2.16](#). Then  $\{U_p \mid p \in I\} \subseteq \mathcal{U}$  is an open covering of all closed points of  $U_s$  and by Hilbert's Nullstellensatz also a covering of  $U_s$ . Since  $U_\eta \in \mathcal{U}$  we get a covering of  $U$  by

$$U = U_\eta \cup \left( \bigcup_{p \in I} U_p \right).$$

$\square$

### 3. DESINGULARIZATION

Let  $S := \operatorname{Spec} R$  be the spectrum of a complete discrete valuation ring with algebraically closed residue field  $k$  and  $X$  be a regular strict semi-stable  $S$ -scheme of dimension 2, i.e., a  $S$ -curve. In this section we describe canonical desingularization methods for two important product situations:



For the first situation let  $K_n/K$  be an algebraic field extension of  $K = \text{Quot } R$  of degree  $n$  and  $R_n$  the ring of integers in  $K_n$ . We will recall a desingularization for the product  $X \times_S \text{Spec } R_n$ . The second situation is the  $k$ -fold product  $X^k$ .

In both cases these desingularizations are well-known: The base change  $X \times_S \text{Spec } R_n$  is well-known for curves ([Del69]). In [Gro95], Gross and Schoen investigate products of regular strict semi-stable schemes in general. The same procedure is later described by Hartl [Har01] together with a desingularization for  $X \times_S \text{Spec } R_n$ , where  $X$  is any regular strict semi-stable scheme. In both techniques the order of the blow-ups used for desingularization is left open and therefore the result is not unique. A different order of blow-ups yields a different scheme and in general a different simplicial reduction set.

Since the order of blow-ups does not matter for curves, we can use for the situation  $X \times_S \text{Spec } R_n$  with  $X$  a  $S$ -curve the known method ([Del69], [Hei03]) unmodified. For the product situation we use an adjusted form of the method described in [Gro95, Prop 6.11] and [Har01, Prop 2.1].

**3.1. Ramified Base-Change of Curves.** We begin with the case of ramified extensions. Let  $K_n/K$  be an algebraic field extension of  $K = \text{Quot } R$  of degree  $n$  and  $R_n$  the ring of integers in  $K_n$ . Let  $X$  be a regular strict semi-stable curve over  $S = \text{Spec } R$ . The base change  $X \times_S \text{Spec } R_n$  is not regular in general, but we can desingularize it in the same way as minimal models are constructed after base change ([Hei03, p. 3]). It turns out that this desingularization turns the reduction graph  $\Gamma$  into the  $n$ -fold subdivision  $\text{sd}_n(\Gamma)$ .

**Theorem 3.1.** *Let  $S := \text{Spec } R$  be the spectrum of a complete discrete valuation ring and  $S_n := \text{Spec } R_n$  the spectrum of the ring  $R_n$  above. Let  $X$  be a regular strict semi-stable  $S$ -curve with a total ordering on  $X^{(0)}$ , whose simplicial reduction set  $\Gamma(X)$  has no multiple simplices. Let  $X_n$  be the scheme obtained by blowing up  $X \times_S S_n$  successively in all singular points, blowing up the resulting scheme successively in all singular points, and so on  $n/2$  times. Then  $X_n$  is a regular strict semi-stable  $S_n$  curve with  $(X_n)_{\eta_n} = (X_\eta) \times_{\text{Spec } K} \text{Spec } K_n$ . Furthermore there exists a total ordering of  $(X_n)^{(0)}$  such that there exists a canonical isomorphism of simplicial reduction sets*

$$\Gamma(X_n) \simeq \text{sd}_n(\Gamma(X)).$$

*Proof.* The desingularization process and the structure of the reduction graph of  $X_n$  is classical, the proof can be done by an explicit calculation like in [Hei03, p. 3]. For the statement about the simplicial reduction sets, we have to define a suitable total ordering on  $(X_n)^{(0)}$ , this can be done in the following way: Since  $\Gamma(X_n)$  and  $\text{sd}_n(\Gamma(X))$  are isomorphic as unordered graphs, there is a bijection  $\varphi : (X_n)^{(0)} \rightarrow \text{sd}_n(\Gamma(X))$  compatible with the graph structure. This means that two components  $C, C' \in (X_n)^{(0)}$  intersect iff there is an edge between the vertices  $\varphi(C), \varphi(C') \in \text{sd}_n(\Gamma(X))_0$ .

By definition A.11 we may identify the elements of  $\text{sd}_n(\Gamma(X))$  with a subset of  $(\Gamma(X)_0)^n = (X_s^{(0)})^n$ . The lexicographical order on  $(X_s^{(0)})^n$  induces then a total ordering on  $\text{sd}_n(\Gamma(X))_0$  and by  $\varphi$  also an total ordering on  $(X_n)^{(0)}$ . With this ordering,  $\varphi$  is also a morphism of simplicial sets. For details see [Kol13, Prop A.27, Lemma 2.6].  $\square$

**3.2. Desingularization of Products.** Let  $X$  be a regular strict semi-stable  $S$ -curve with total ordering on  $X^{(0)}$ . The product  $X^d := X \times_S \cdots \times_S X$  is in general not regular strict semi-stable. This can already be observed with the standard scheme  $X = L := \text{Spec } R[x_0, x_1]/(x_0x_1 - \pi)$ . We use a desingularization similar to [Gro95, Prop 6.11] and [Har01] to get a regular strict semi-stable model  $W(X, <, d)$

of  $(X_\eta)^d$ . The result of this process depends on a sequence of components, which is chosen arbitrarily in the citations above. Using the total ordering on  $X^{(0)}$  we are able to define a canonical sequence and thus get a result with a well-defined simplicial reduction set.

The desingularization works as follows:

**Algorithm 3.2.** Let  $d \in \mathbb{N}$  and  $X$  be a regular strict semi-stable  $S$ -curve with total ordering  $\leq$  on  $X_s^{(0)}$  and  $\Gamma(X)$  a simplicial set without multiple simplices. We denote the product by  $W_0 := X^d$ . Since the components of  $X_s$  are geometrically integral, we can describe the irreducible components of  $(W_0)_s$  as product

$$(W_0)_s^{(0)} = X_s^{(0)} \times \cdots \times X_s^{(0)}.$$

We endow this product  $(W_0)_s^{(0)}$  with the lexicographical order and denote the elements in ascending order  $B_1, \dots, B_k$ . Now denote by  $B'_1$  the irreducible component  $B_1$  endowed with the induced reduced structure and set  $W_1 := \text{Bl}_{B'_1}(W_0)$ . Inductively let  $B'_i \subseteq W_i$  be the strict transform of the irreducible component  $B_i$  endowed with the induced reduced structure and set  $W_{i+1} := \text{Bl}_{B'_i}(W_i)$ . The last scheme in this chain,  $W_k$ , is also denoted by  $W(X, \leq, d) := W_k$ . These blowups introduce no new components in the special fibre  $(W_k)_s$ , so the lexicographical ordering on  $(W_0)_s^{(0)}$  also induces a total ordering on  $(W_k)_s^{(0)}$ .

**Theorem 3.3.** The scheme  $W(X, \leq, d)$  constructed in [algorithm 3.2](#) is regular strict semi-stable and the reduction set with respect to the lexicographical ordering induced by  $(W_0)_s^{(0)}$  is given by  $\mathcal{R}(W) = \Gamma(X)^d$ .

Before we proof this fact, let us first assume that  $X$  is the standard scheme  $X = L := \text{Spec } R[x_0, x_1]/(x_0x_1 - \pi)$ . In this case we may describe all schemes  $W_i$  in [algorithm 3.2](#) explicitly by an affine covering. In this covering we are able to check the claims easily. The covering schemes are spectra of the following algebras:

**Definition 3.4.** Let  $l \in \mathbb{N}_0$  be a natural number and  $A$  a non-empty set. We define a  $R$ -algebra  $M(l, A)$  by

$$M(l, A) := R[w_1, \dots, w_l][u_{a,0}, u_{a,1} \mid a \in A]/\mathcal{I}_M,$$

the quotient of a free commutative algebra generated by the elements  $w_1, \dots, w_l$  and  $\{u_{a,0}, u_{a,1} \mid a \in A\}$  by the Ideal  $\mathcal{I}_M$ , which is generated by

$$\mathcal{I}_M := \{w_1 \cdots w_l u_{a,0} u_{a,1} - \pi, u_{a,0} u_{a,1} - u_{a',0} u_{a',1} \mid a, a' \in A\}.$$

We first analyse the irreducible components and blow-ups of  $\text{Spec } M(l, A)$ .

**Lemma 3.5.** Let  $l \in \mathbb{N}$  and  $A$  be a finite set.

- (i) The affine scheme  $N := \text{Spec } M(l, A)$  is integral. Its special fibre  $N_s$  consists of the following components: For each  $i \in \{1, \dots, l\}$  a Cartier divisor  $D(i)$  given by the ideal  $(w_i)$  and for each mapping  $t \in \text{Hom}(A, \{0, 1\})$  a component  $C(t)$  given by the ideal  $(u_{k,t(a)} \mid a \in A)$ .
- (ii) The blow-up  $\tilde{N} := \text{Bl}_{C(0)}(N)$  of  $N$  at the component  $C(0)$  corresponding to the zero mapping  $0 : A \rightarrow \{0, 1\}$  is covered by the family of affine schemes

$$(\tilde{N}_a := \text{Spec}(M(l+1, A \setminus \{a\})))_{a \in A}.$$

- (iii) For each chart  $\tilde{N}_a$  the irreducible components of  $(\tilde{N}_a)_s$  have the form  $\tilde{D}_i$  with  $i \in \{1, \dots, l+1\}$  or  $\tilde{C}(t)$  with  $t \in \text{Hom}(A \setminus \{a\}, \{0, 1\})$  analogous to (i). Under the blow-up morphism  $\psi_a : \tilde{N}_a \rightarrow N$  these components are mapped onto the

following components of  $N_s$ :

$$\begin{aligned}\psi_a(\tilde{D}(i)) &= D(i) \quad \text{for } i \in \{1, \dots, l\}, \\ \psi_a(\tilde{D}(l+1)) &= C(0), \\ \psi_a(\tilde{C}(t|_{A \setminus \{a\}})) &= C(t) \quad \text{for } t \in \text{Hom}(A, \{0, 1\}) \text{ with } t(a) = 1.\end{aligned}$$

(iv) Let  $\psi_a$  be defined as above and denote by  $E \in \text{CaDiv}(\tilde{N})$  the exceptional divisor of the blow-up. Then we have for any  $\alpha \in A \setminus \{a\}$  the following identities of principal Cartier divisors:

$$\begin{aligned}\psi_a^{-1}(\text{div}(u_{a,0}))E^{-1}|_{\tilde{N}_a} &= \text{div}(1), \\ \psi_a^{-1}(\text{div}(u_{a,1}))|_{\tilde{N}_a} &= \text{div}(\tilde{u}_{\alpha,0}\tilde{u}_{\alpha,1}), \\ \psi_a^{-1}(\text{div}(u_{\alpha,0}))E^{-1}|_{\tilde{N}_a} &= \text{div}(\tilde{u}_{\alpha,0}), \\ \psi_a^{-1}(\text{div}(u_{\alpha,1}))|_{\tilde{N}_a} &= \text{div}(\tilde{u}_{\alpha,1}).\end{aligned}$$

*Proof.* All these claims can be shown by explicite calculations just using [Liu02, Lemma 8.1.4]. For details see [Kol13, Lemma 2.14].  $\square$

Before we are able to return to the proof of [theorem 3.3](#) with  $X = L$  we need to introduce some additional notation. On  $W_0 := L^d$  we denote by  $\text{pr}_i : W_0 \rightarrow L$  the projection on the  $i$ -th factor. We denote the irreducible components of  $L_s$  by  $I_0, I_1$  with the obvious ordering  $I_0 < I_1$ . Each component  $C$  of  $(W_0)_s$  is of the form  $C = I_{t(1)} \times \dots \times I_{t(d)}$  for a mapping  $t \in \text{Hom}(\{1, \dots, d\}, \{0, 1\})$ . We denote this mapping by  $t_C$ .

To apply [algorithm 3.2](#) we arrange the components of  $W_s$  lexicographically as  $B_1, \dots, B_{2^d}$ . In contrast to that we will also use the product order on  $(W_s)^{(0)}$ , which will be denoted by  $\leq$ : For two components  $C, C' \in (W_s)^{(0)}$  set  $C \leq C'$  iff for each  $i$  the relation  $t_C(i) \leq t_{C'}(i)$  holds. We denote by  $J := \{C_1 < \dots < C_{l+1}\} \subseteq \mathcal{P}(\mathcal{R}(W_0)_0)$  the set of all strictly increasing chains with respect to  $\leq$  which cannot be refined. For each  $m \in \mathbb{N}_0$  we define the subset  $J^m$  by

$$J^m := \{(C_1 < \dots < C_{l+1}) \in J \mid C_1 = B_1, \{C_1, \dots, C_l\} \subseteq \{B_1, \dots, B_m\}\}.$$

Thus elements of  $J^m$  are chains of components, which cannot be refined and whose elements – with the exception of the last element – are already blown-up in step  $m$ . In particular one gets  $J^0 = \{(B_1)\}$ .

**Lemma 3.6.** *Let  $X = L$  and  $W_m$  constructed as in [algorithm 3.2](#). Then for each  $m \in \{0, \dots, 2^d\}$  the following holds:*

(i) *The scheme  $W_m$  is covered by the family of affine schemes*

$$(3.1) \quad \left( \text{Spec } M(l, A(C_{l+1})) \right)_{(C_1 < \dots < C_{l+1}) \in J^m}$$

where  $A(C_l)$  denotes the subset  $A(C) := t_C^{-1}(1) \subseteq \{1, \dots, d\}$ . The irreducible components of  $(W_m)_s$  are exactly the strict transforms of the components of  $(W_0)_s$ .

(ii) *In the chart  $N(C_1, \dots, C_{l+1}) := \text{Spec } M(l, A(C_{l+1}))$  corresponding to the chain  $(C_1 < \dots < C_l < C_{l+1})$  the components  $C_1, \dots, C_l$  are given by the Cartier divisors  $D(1), \dots, D(l)$ . The chart also contains the strict transforms of all components  $C \in (W_0)_s^{(0)}$  with  $C \geq C_l$ . Let  $C \geq C_l$  be such a component, then the strict transform of  $C$  is given in the chart by  $C(t_C|_{A(C_k)})$ .*

(iii) Each centre of blow-up is the scheme-theoretic intersection of Cartier divisors. In particular the centre  $B'_j$  is given by

$$B'_j = \bigcap_{n=1}^d F_{j,n}$$

with the Cartier divisors

$$F_{j,n} := \text{pr}_n^{-1}(\text{pr}_n(B_j)) - \sum_{i \leq m, \text{pr}_n(B_i) = \text{pr}_n(B_j)} B_i.$$

*Proof.* We show (i) and (ii) by induction on  $m$ . The claim is trivial for  $m = 0$ : We have  $J^0 = \{B_1\}$  and  $L^d$  is indeed covered by the affine scheme  $\text{Spec } M(0, \{1, \dots, d\}) \simeq L^d$ . The isomorphism can be chosen in such a way that for each  $n \in \{1, \dots, d\}$  the projection  $\text{pr}_n$  induces a morphism on the global sections

$$\mathcal{O}_L(L) \rightarrow \mathcal{O}_{N(B_1)}(N(B_1)),$$

which maps  $x_0$  onto  $u_{n,0}$  and  $x_1$  onto  $u_{n,1}$ . Then (ii) follows immediately.

Assume now that (i) and (ii) are true for  $m \geq 0$ . We have to examine the blow-up  $B_{m+1} = \text{Bl}_{B'_{m+1}}(W_m)$  and can restrict ourself to the charts of  $W_m$  which contain the component  $B'_{m+1}$ . This means we consider chains  $(C_1 < \dots < C_k) \in J^m$  with  $C_k = B_{m+1}$  with their associated chart  $N = N(C_1, \dots, C_k)$ . Since  $t_{C_k}|_{A(C_k)} = 0$  we can apply [lemma 3.5](#) to describe the blow-up:

The blow-up  $\tilde{N}$  of  $N$  in  $C_k$  is covered by the charts

$$\text{Spec } M(k+1, A(C_k) \setminus \{l\}), l \in A(C_k).$$

If we associate to each  $l \in A(C_k)$  the component  $C'$  with  $A(C') = A(C) \setminus \{l\}$  and hence an element  $(C_0 < \dots < C_l < C') \in J^{m+1}$ , this proves (i). By [lemma 3.5](#) again a component  $C'' > C$  lies in the chart  $N(C_0, \dots, C_l < C')$  iff  $t_{C''}(a) = 1$  holds, which is equivalent to  $C'' > C'$ .

For the description of the centres of blow-up we show inductively using [lemma 3.5](#) that in each chart  $N(C_1, \dots, C_{l+1}) \setminus W_m$  the following identities hold:

$$F_n|_{N(C_1, \dots, C_{l+1})} = \begin{cases} V(u_{n,0}) & \text{if } \text{pr}_n(C) = 0, \text{pr}_n(C_l) = 0, \\ V(1) & \text{if } \text{pr}_n(C) = 0, \text{pr}_n(C_l) = 1, \\ V(u_{n,1}) & \text{if } \text{pr}_n(C) = 1, \text{pr}_n(C_l) = 0, \\ V(u_{a',0}u_{a',1}) \text{ for any } a' \in A(C_l) & \text{if } \text{pr}_n(C) = 1, \text{pr}_n(C_l) = 1. \end{cases}$$

If the blow-up centre  $B_j$  suffices  $B_j > C_k$ , we get

$$\bigcap_{n=1}^d F_{j,n}|_{N(C_1, \dots, C_{l+1})} = V((u_{a,t_{B_j}(a)} \mid a \in A(C_l)) = B_j.$$

If otherwise  $B_j \not> C_k$ , we have

$$\bigcap_{n=1}^d F_{j,n}|_{N(C_1, \dots, C_{l+1})} = \emptyset.$$

The last claim then follows directly.  $\square$

*Proof of [theorem 3.3](#).* First assume  $X = L$ . By [lemma 3.6](#) the scheme  $W(L, <, d) := W_{2^d}$  is covered by affine charts of the form

$$\text{Spec } M(k, \{a\}) \simeq \text{Spec } R[w_1, \dots, w_l, w_{l+1}, w_{l+2}] / (w_1 \cdots w_{l+2} - \pi)$$

and is therefore regular strict semi-stable according to [theorem 2.3](#).

It remains to find an isomorphism of simplicial sets  $\mathcal{R}(W_{2^d}) \simeq (\mathcal{R}(L))^d$ . [proposition 2.9](#) ensures that the reduction sets of  $L$  and  $W_{2^d}$  contain no multiple simplices,

thus by [proposition 2.13](#) the projection morphisms  $\tilde{\text{pr}}_i : M_m \rightarrow L$  induce morphisms  $(\tilde{\text{pr}}_i)_* : \mathcal{R}(W_{2^d}) \rightarrow \mathcal{R}(L)$ . By the universal property of the product they induce a morphism  $\mathcal{R}(W_{2^d}) \rightarrow \mathcal{R}(L)^d$  and we have to show injectivity and surjectivity. On the level of 0-simplices this is trivial, since the blow-up morphism  $W_{2^d} \rightarrow W_0 = L^d$  do not induce additional components in the special fibre, thus  $(W_{2^d})_s^{(0)} \simeq (W_0)_s^{(0)} \simeq (L_s^{(0)})^d$ .

To show injectivity and surjectivity for  $k$ -simplices ( $k > 0$ ) it suffices by [proposition 2.9](#) that the following two sets coincide:

$$\left\{ (C_0 \leq \dots \leq C_k) \in ((M_m)^{(0)})^{k+1} \mid C_0 \cap \dots \cap C_k \neq \emptyset \right\},$$

$$\left\{ (C_0 \leq \dots \leq C_k) \in ((M_m)^{(0)})^{k+1} \mid \tilde{\text{pr}}_i(C_0) \cap \dots \cap \tilde{\text{pr}}_i(C_k) \neq \emptyset \quad \forall i \in \{1, \dots, d\} \right\}.$$

Let  $(C_0 \leq \dots \leq C_k) \in (W_{2^d})^{(0)}$ . The components  $C_0, \dots, C_k$  intersect iff they are contained in a common chart, which means they have to be totally ordered by the product order defined on  $W_{2^d}^{(0)}$ . This is in turn equivalent to  $\tilde{\text{pr}}_i(C_0) \cap \dots \cap \tilde{\text{pr}}_i(C_k) \neq \emptyset$  for all  $i$ .

The general case, where  $X$  is any regular strict semi-stable curve, can be reduced to the local case above: The property of  $W(X, \leq, d)$  to be regular strict semi-stable can be checked locally. By [proposition 2.14](#) also the simplicial reduction set can be determined in an open covering. Thus we may assume by [corollary 2.17](#) that there is a smooth map  $X \rightarrow L$ . By the commutativity of blow-ups with flat base change ([\[Liu02, Prop 8.1.12\]](#)) it suffices to deal with the case  $X = L$ . For details see [\[Kol13, Lemma 2.11\]](#).  $\square$

From [lemma 3.5\(iv\)](#) we immediately deduce:

**Corollary 3.7.** *The centres of all blow-up morphisms in [algorithm 3.2](#) are scheme-theoretic intersections of Cartier divisors.*

#### 4. INTERSECTION NUMBERS FOR PRODUCTS

After the construction of regular strict semi-stable models on products of semi-stable curves, we use the combinatorial structure of the reduction set to calculate intersection numbers. Let  $S$  be the spectrum of a discrete valuation ring with algebraically closed residue field and  $X$  a regular strict semi-stable  $S$ -curve with a total ordering  $\leq$  on  $X^{(0)}$ . The reduction graph is denoted by  $\Gamma := \Gamma(X)$ . For a chosen  $d \in \mathbb{N}$  we examine the product model  $W := W(X, \leq, d)$  of  $(X_\eta)^d$  constructed according to [algorithm 3.2](#). By [theorem 3.3](#) the simplicial reduction set is  $\mathcal{R}(W) = \Gamma^d$ . Some essential relations in the Chow ring  $\text{CH}_{W_s}(W)$  depend only on the combinatorial structure of  $\mathcal{R}(W)$ . We define a ring  $\mathcal{C}(\Gamma^d)$  called combinatorial Chow ring, which encodes these relations. A moving lemma in  $\mathcal{C}(\Gamma^d)$  enables us to calculate intersection numbers in a more localised way. The intersection numbers in  $\mathcal{C}(\Gamma^d)$  for any graph  $\Gamma$  can therefore be calculated only with the knowledge of  $\mathcal{C}(I^d)$ , where  $I$  is the graph which consists of exactly one edge. For such a graph we can eventually give explicit calculations of intersection numbers in the case  $d = 2$  and  $d = 3$ .

**4.1. Intersection Theory on Regular Schemes.** For our calculations we need only the basic facts of intersection theory, i.e., intersection with Cartier divisors. This is covered in [\[Ful98, Chapter 1,2\]](#) if one applies the generalizations from Chapter 20. We give a short explanation of the notation, recall the basic facts and deduce a lemma considering ramified base change.

**Definition 4.1.** Let  $S$  be a regular scheme and  $X$  be a  $S$ -scheme of finite type with structure morphism  $\varphi : X \rightarrow S$ .

- (i) If  $X$  is irreducible with generic point  $\eta_X$  we define the *relative dimension* of  $X$  over  $S$  by:

$$\dim_S(X) := \text{trdeg}(\kappa(\eta_X)/\kappa(\varphi(\eta_X))) - \dim(\mathcal{O}_{S, \varphi(\eta_X)}).$$

- (ii) If  $X$  is any  $S$ -scheme of finite type it is called *relative equidimensional of dimension  $d$* , if for each irreducible component  $V$  the equation

$$\dim_S(V) = d$$

holds. In this case we call  $d$  the relative dimension of  $X$  with respect to  $S$ .

**Definition 4.2.** Let  $S$  be a regular Noetherian scheme and  $X$  a regular  $S$ -scheme of finite type, which is relative equidimensional of dimension  $d = \dim_S(X)$ . Let  $Y \subseteq X$  be a closed subset and  $p \in \mathbb{N}$  be a number. We denote by  $\text{CH}_Y^p(X)$  the  $p$ -th Chow group with support in  $Y$ , i.e., in the notation of [Ful98]

$$\text{CH}_Y^p(X) := A_{d-p}(Y).$$

*Remark 4.3.* Let  $X, X'$  be regular relative equidimensional schemes and  $f : X' \rightarrow X$  a morphism. Let  $Z \subseteq X, Z' \subseteq X'$  be closed subschemes.

- (i) If  $f$  is flat, it induces a morphism  $f^* : \text{CH}_Z^p(X) \rightarrow \text{CH}_{f^{-1}(Z)}^p(X')$  for each  $p \in \mathbb{N}$ .
- (ii) If  $f$  is any morphism, there exists a morphism  $f^* : \text{CH}_Z^1(X) \rightarrow \text{CH}_{f^{-1}(Z)}^1(X')$ .
- (iii) If  $f$  is proper, it induces a morphism  $f_* : \text{CH}_Z^p(X') \rightarrow \text{CH}_{f(Z)}^{p-d}(X)$ , where  $d$  is the relative dimension of  $X'$  over  $X$ .

All these constructions are functorial.

*Proof.* These are basic facts [Ful98, 1.7, 2.2, 1.4]. For the second claim, note that  $\text{CH}_Y^1(X)$  coincides with the pseudo-divisors on  $X$  with support  $Y$ .  $\square$

*Remark 4.4.* Fulton defines an intersection product for divisors using (ii): Let  $X$  be a regular scheme. Then there is the intersection product

$$\cdot : \text{CH}_Y^1(X) \otimes \text{CH}_Z^p(X) \rightarrow \text{CH}_{Y \cap Z}^{p+1}(X)$$

defined for each divisor  $D \in \text{CH}_Y^1(X)$  and each integral closed subscheme  $V$  by

$$D \cdot [V] = [j^* D]$$

where  $j : V \rightarrow X$  denotes the canonical inclusion.

We will only consider elements of  $\text{CH}_{X_s}^p(X)$  which are products of elements from  $\text{CH}_{X_s}^1(X)$ . [Ful98, Prop 2.3 (d)] suggests the following generalization of the pull-back:

**Definition 4.5.** Let  $\alpha = D_1 \cdots D_p \in \text{CH}_{X_s}^p(X)$  be the product of Cartier divisors  $D_1, \dots, D_p \in \text{CH}_{X_s}^1(X)$  and  $f : X' \rightarrow X$  any morphism. Then  $f^* \alpha$  is defined as

$$f^* \alpha = (f^* D_1) \cdots (f^* D_p).$$

Let  $S = \text{Spec } R$  be the spectrum of a complete discrete valuation ring with algebraic closed residue field and  $X$  a regular proper  $S$ -scheme. By a construction similar to [Ful98, Def 1.4] we define a local degree for cycles with support in the special fibre  $X_s$ . For this definition we remark that the special fibre of  $S$  consists only of one point  $\{s\}$  which has codimension 1, therefore we have a canonical isomorphism  $\text{CH}_{\{s\}}^1(\text{Spec } R) \simeq \mathbb{Z}$ .

**Definition 4.6.** Let  $X$  be a flat proper  $S$ -scheme of relative dimension  $d$  with structure morphism  $f : X \rightarrow S$ . The morphism

$$\text{ldeg}_X := f_* : \text{CH}_{X_s}^{d+1}(X) \rightarrow \text{CH}_{\{s\}}^1(S) \simeq \mathbb{Z}$$

is called *local degree*.

We are now able to describe the behaviour of the intersection product at ramified base change: Let  $K_n/K$  be an algebraic extension of degree  $n$  of  $K = \text{Quot}(R)$ , let  $R_n$  be the ring of integers in  $K_n$  and  $S_n := \text{Spec } R_n$  be the spectrum of  $R_n$ . Since the structure morphism  $g : S_n \rightarrow S$  is proper, we can consider the direct image  $g_* : \text{CH}(S_n) \rightarrow \text{CH}(S)$ . This enables us to compare intersection multiplicities on  $S$ -schemes and  $S_n$ -schemes:

**Lemma 4.7.** *Let  $W$  be a integral flat  $S$ -scheme and  $W_n$  an integral  $S_n$ -scheme with a proper  $S$ -morphism  $\varphi : W_n \rightarrow W$ . Assume that  $\varphi|_{(W_n)_\eta}$  is flat and  $(W_n)_\eta = W_\eta \times_{S_\eta} (S_n)_\eta$  holds. Then for each  $\alpha = D_1 \cdots \cdots D_d \in \text{CH}_{W_s}^d(W)$  with  $D_1, \dots, D_d \in \text{CH}_{W_s}^1(W)$  the equation*

$$\varphi_*(\varphi^*(\alpha)) = n\alpha$$

*holds in  $\text{CH}_{W_s}^d(W)$ . If furthermore  $W$  is a proper  $S$ -scheme and  $W_n$  a proper  $S_n$ -scheme, then for  $d = \dim W$  and each  $\alpha \in \text{CH}_{W_s}^d(W)$  we get the equation*

$$\text{ldeg}_W(\alpha) = n \text{ldeg}_{W_n}(\varphi^*(\alpha)).$$

*Proof.* We first show  $\varphi_*([W_n]) = n[W]$ . The image  $\varphi(W_n)$  is irreducible, hence  $\varphi_*([W_n])$  is a multiple of  $[W]$ . To determine the multiplicity we restrict ourself to the generic fibre and consider the Cartesian square

$$\begin{array}{ccc} (W_n)_{\eta_n} & \xrightarrow{f'} & (S_n)_{\eta_n} \\ \varphi \downarrow & & g \downarrow \\ W_\eta & \xrightarrow{f} & S_\eta. \end{array}$$

Using  $g_*([(S_n)_{\eta_n}]) = n[S_\eta]$  we get

$$\varphi_*([W_n]_{\eta_n}) = \varphi_*(f'^*([(S_n)_{\eta_n}])) = f^*(g_*([(S_n)_{\eta_n}])) = nf^*([S_\eta]) = n[W_\eta].$$

Let now  $\alpha = D_1 \cdots \cdots D_d \in \text{CH}_{W_s}(W)$  be arbitrary. Then the claim is proven using the projection formula [Ful98, Prop 2.3 (c)]:

$$\varphi_*(\varphi^*(\alpha)) = \varphi_*(\varphi^*(\alpha) \cdot [W_n]) = \alpha \cdot \varphi_*([W_n]) = n\alpha.$$

For the second claim consider the commutative diagram

$$\begin{array}{ccc} (W_n) & \xrightarrow{f'} & (S_n) \\ \varphi \downarrow & & g \downarrow \\ W & \xrightarrow{f} & S, \end{array}$$

where  $f$  and  $f'$  are proper. By definition of the degree map we get

$$f_*\varphi_*(\varphi^*\alpha) = nf_*\alpha = n \text{ldeg}_W(\alpha)[\{s\}]$$

and

$$f_*\varphi_*(\varphi^*\alpha) = g_*f'_*(\varphi^*\alpha) = \text{ldeg}(\varphi^*\alpha)g_*([\{s_n\}]) = \text{ldeg}(\varphi^*\alpha)[\{s\}].$$

The claim now follows by equating coefficients.  $\square$



**4.2. The Chow Ring of Product Models.** We use the facts we recalled so far to study the Chow ring with support in the special fibre of regular strict semi-stable schemes. In particular we consider the product models constructed in [algorithm 3.2](#). Let as usual  $X$  be a regular strict semi-stable curve over  $S$  with total ordering  $\leq$  on  $X_s^{(0)}$ . We assume that the reduction graph  $\Gamma(X)$  has no multiple edges. Furthermore let  $W := W(X, \leq, d)$  denote the desingularization of  $X^d$  constructed in [algorithm 3.2](#). We denote the composition of the desingularization map  $W \rightarrow X^d$  with the projection on the  $i$ th factor by  $\text{pr}_i : W \rightarrow X$ . Since  $\Gamma(X)$  is without multiple simplices, this is also true for  $\mathcal{R}(W)$ . We may deduce the following relations in the Chow ring of  $W$ :

**Proposition 4.8.** *Let  $C_1, \dots, C_k$  be pairwise different irreducible components of  $W_s$  with  $C_1 \cap \dots \cap C_k \neq \emptyset$ . Then these components intersect properly and with multiplicity 1, i.e.,*

$$[C_1] \cdots [C_l] = [C_1 \cap \dots \cap C_l].$$

*If furthermore  $l = d$  we have*

$$\text{ldeg}([C_1] \cdots [C_l]) = 1.$$

*Proof.* The components  $C_1, \dots, C_k$  intersect properly according to [definition 2.1](#) and by [proposition 2.9](#) the intersection  $C_1 \cap \dots \cap C_i$  is irreducible. Using induction, it is enough to show  $\chi^p(C_1 \cap \dots \cap C_{l-1}, C_l) = 1$  in the generic point  $p$  of  $C_1 \cap \dots \cap C_l$  where  $\chi^p$  denotes the intersection multiplicity of Serre. This is a direct consequence of the following variant of the criterion for multiplicity 1 of Fulton [[Ful98](#), 7.2]:  $\square$

**Lemma 4.9.** *Let  $Y, Z$  be integral regular closed subschemes of a regular scheme  $X$  of codimension  $p$  resp.  $q$ , which have proper intersection. If the scheme-theoretic intersection  $Y \cap Z$  is reduced, then  $Y \cap Z$  is regular and Serre's intersection multiplicity gives*

$$\chi^x(Y, Z) = 1$$

*for each generic point  $x$  of an irreducible component of  $Y \cap Z$ .*

*Proof.* Let  $x$  be the generic point of an irreducible component of  $Y \cap Z$ . Denote the local ring  $\mathcal{O}_{X,x}$  with  $A$  and the defining ideals for  $Y$  and  $Z$  with  $\mathcal{I}_Y$  resp.  $\mathcal{I}_Z$ . Since  $Y$  is regular,  $\mathcal{I}_Y$  is generated by a regular sequence  $(y_1, \dots, y_p)$  of length  $p$ ; likewise the ideal  $\mathcal{I}_Z$  is generated by a regular sequence  $(z_1, \dots, z_q)$ . Since  $x$  is a generic point, the elements  $(y_1, \dots, y_p, z_1, \dots, z_q)$  constitute an generating set for the maximum ideal  $\mathfrak{m}_{X,x}$  of  $\mathcal{O}_{X,x}$ . Therefore  $\mathcal{O}_{X,x}$  is itself regular and the generating set is a system of parameters.

By [[Ser00](#), IV A Cor. 2] we may use the Koszul-complex to calculate

$$\text{Tor}_i(A/(\mathcal{I}_Y)_x, A/(\mathcal{I}_Z)_x) \simeq H_i((y_1, \dots, y_p), A/(\mathcal{I}_Z)_x)$$

and eventually by [[Ser00](#), IV A Prop. 3]

$$H_i((y_1, \dots, y_p), A/(\mathcal{I}_Z)_x) = 0 \text{ for all } i \geq 1.$$

Therefore we have

$$\chi^x(Y, Z) = \text{length}(A/(\mathcal{I}_Y)_x \otimes A/(\mathcal{I}_Z)_x) = \text{length}(A/\mathfrak{m}) = 1.$$

$\square$

**Proposition 4.10.** *For each irreducible component  $C \in W_s^{(0)}$  of  $W_s$  the equation*

$$\left( \sum_{C' \in W_s^{(0)}} [C'] \right) \cdot [C] = 0 \in \text{CH}_{W_s}(W)$$

*holds.*

*Proof.* Since  $W_s$  is reduced, we have  $\sum_{C' \in W_s^{(0)}} [C'] = \text{div}(\pi) \in \text{Rat}_{W_s}(W)$ . Therefore the intersection product with the cycle  $[C]$  vanishes in the Chow ring  $\text{CH}_{W_s}(W)$ .  $\square$

**Proposition 4.11.** *Let  $C_1, C_2 \in \mathcal{R}(W)_0$  be irreducible components of  $W_s$ . If there is an  $i \in \{1, \dots, d\}$  such that  $\text{pr}_i(C_1) \neq \text{pr}_i(C_2)$ , then the equation*

$$[C_1] \cdot [C_2] \cdot \left( \sum_{\substack{C' \in \mathcal{R}(W)_0, \\ \text{pr}_i(C') = \text{pr}_i(C_2)}} [C'] \right) = 0$$

*holds in  $\text{CH}_{W_s}(W)$ .*

*Proof.* We can assume that the intersection of  $C_1$  and  $C_2$  is non-empty. Since  $\mathcal{R}(X)$  has no multiple simplices and  $\text{pr}_i(C_1) \neq \text{pr}_i(C_2)$ , the projections  $\text{pr}_i(C_1)$  and  $\text{pr}_i(C_2)$  have to intersect in one double point. We denote this point by  $p \in X$ . The component  $\text{pr}_i(C_2)$  of  $X_s$  is given by a Cartier divisor  $\tilde{C} \in \text{CaDiv}(X)$  and can be represented by a section  $r \in \Gamma(U, \mathcal{O}_X)$  in an open neighbourhood  $U$  of  $p$ . As the scheme  $X$  is integral, we can continue  $r$  as a rational function  $r \in \Gamma(X, \mathcal{K}_X)$  and we have  $\text{div}(r) = \tilde{C} + \tilde{C}_r$ , where  $\tilde{C}_r$  is a Cartier divisor with support outside of  $U$ . Using the pull-back we get a principal divisor on  $W$ , which splits into  $\text{pr}_i^*(\text{div}(r)) = \text{pr}_i^*(\tilde{C}) + \text{pr}_i^*(\tilde{C}_r)$ . Since  $X_s$  is geometrically reduced, we have

$$[\text{pr}_i^*(\tilde{C})] = \sum_{\substack{C' \in \mathcal{R}(W)_0, \\ \text{pr}_i(C') = \text{pr}_i(C_2)}} [C'],$$

while the residue divisor  $\text{pr}_i^*(\tilde{C}_r)$  has support outside of  $p$  and does not contribute to the intersection product  $[C_1][C_2]$ . Calculating in  $\text{CH}_{W_s}(W)$  we finally get

$$0 = [C_1][C_2][\text{pr}_i^*(\text{div}(r))] = [C_1][C_2] \sum_{\substack{C' \in \mathcal{R}(W)_0, \\ \text{pr}_i(C') = \text{pr}_i(C_2)}} [C'].$$

$\square$

**4.3. A Moving Lemma in the Special Fibre.** The rational equivalences considered in [proposition 4.10](#) and [proposition 4.11](#) depend only on the simplicial reduction set of the model  $W$  and on the projections  $\text{pr}_i : \mathcal{R}(W) \rightarrow \mathcal{R}(X)$ . This observation leads us to define a combinatorial Chow ring, which describes the part of the Chow ring which is independent of the concrete model. On this Chow ring we can proof a moving lemma, which is used later to calculate intersection numbers.

For this section let  $d \in \mathbb{N}$  be an integer and  $\Gamma$  a finite ordered graph without multiple edges, i.e. a simplicial set of dimension 1 without multiple simplices. We denote by  $\Gamma^d$  the product of simplicial sets according to [remark A.6](#) and by  $\text{pr}_i : \Gamma^d \rightarrow \Gamma$  the projection on the  $i$ th component. Furthermore we use the notation from [definition A.9](#), where  $(\Gamma^d)_S \subseteq \mathcal{P}((\Gamma^d)_0)$  denotes the family of all subsets, which occur as node set of a simplex in  $\Gamma^d$ , i.e., the set

$$(\Gamma^d)_S = \{\{\sigma(0), \dots, \sigma(n)\} \mid n \in \mathbb{N}, \sigma \in (\Gamma^d)_n\}.$$

**Definition 4.12.** We denote with  $Z(\Gamma^d)$  the polynomial ring  $Z(\Gamma^d) := \mathbb{Z}[C \mid C \in (\Gamma^d)_0]$  generated by the 0-simplices. It is supplied with the usual grading, which gives all generators  $C \in (\Gamma^d)_0$  the degree 1.

We define a graded ideal  $\text{Rat}(\Gamma^d)$  on  $Z(\Gamma^d)$  generated by the polynomials

$$(4.1) \quad C_1 \cdots C_k \quad \text{for } \{C_1, \dots, C_k\} \notin (\Gamma^d)_S,$$

$$(4.2) \quad \left( \sum_{C' \in (\Gamma^d)_0} C' \right) C_1 \quad \forall C_1 \in (\Gamma^d)_0,$$

$$(4.3) \quad \sum_{\substack{C' \in (\Gamma^d)_0 \\ \text{pr}_i(C') = \text{pr}_i(C_2)}} C_1 C_2 C' \quad \forall C_1, C_2 \in (\Gamma^d)_0, i \in \{1, \dots, d\} \\ \text{with } \text{pr}_i(C_1) \neq \text{pr}_i(C_2).$$

We call  $\text{Rat}(\Gamma^d)$  the ideal of cycles rationally equivalent to zero.

The graded ring

$$\mathcal{C}(\Gamma^d) := Z(\Gamma^d) / \text{Rat}(\Gamma^d)$$

is called *combinatorial Chow ring*.

The combinatorial Chow ring has the following functoriality:

**Proposition 4.13.** *Let  $\Gamma$  and  $\Gamma'$  be finite ordered graphs without multiple simplices and  $f_1, \dots, f_d : \Gamma' \rightarrow \Gamma$  morphisms of graphs. Let  $f = (f_1, \dots, f_d) : (\Gamma')^d \rightarrow \Gamma^d$  be the induced morphism on the products. Then we have a well-defined morphism of rings given by*

$$f^* : Z(\Gamma^d) \rightarrow Z(\Gamma'^d), C \mapsto \sum_{\substack{C' \in (\Gamma'^d)_0, \\ f(C') = C}} C'.$$

*It induces an homomorphism of combinatorial Chow rings*

$$f^* : \mathcal{C}(\Gamma^d) \rightarrow \mathcal{C}(\Gamma'^d).$$

*Proof.* We have to show  $f^*(\text{Rat}(\Gamma^d)) \subseteq \text{Rat}(\Gamma'^d)$  and can restrict ourself to the generators (4.1), (4.2) and (4.3).

For each set  $\alpha \in (\Gamma^d)_S$  we have  $f(\alpha) \in (\Gamma'^d)_S$ .

Consider a polynomial of type (4.1), i.e., a product  $\alpha = C_1 \cdots C_k$  with  $\{C_1, \dots, C_k\} \notin (\Gamma^d)_S$ . Then each monomial of  $f^*\alpha$  has the form  $\tilde{C}_1 \cdots \tilde{C}_k$  with  $f(\tilde{C}_i) = C_i$  and since  $f$  is a morphism of simplicial sets,  $\{\tilde{C}_1, \dots, \tilde{C}_k\} \notin (\Gamma'^d)_S$  holds. This implies  $f^*(C_1 \cdots C_k) \in \text{Rat}(\Gamma'^d)$ .

Similarly, one can show the claim for elements of type (4.2) and (4.3) by means of the equations

$$f^* \left( \sum_{C' \in ((\Gamma')^d)_0} C' \right) = \sum_{C \in (\Gamma^d)_0} C$$

and

$$f^* \left( \sum_{\substack{C' \in ((\Gamma')^d)_0 \\ \text{pr}_i(C') = \text{pr}_i(C'_2)}} C' \right) = \sum_{\substack{C \in (\Gamma^d)_0 \\ \text{pr}_i(C) = \text{pr}_i(f(C'_2))}} C.$$

□

With the definition of the combinatorial Chow ring we can sum up our knowledge about the Chow ring of a product model  $W = W(X, <, d)$  by:

**Proposition 4.14.** *Let  $d \in \mathbb{N}$ ,  $X$  a regular strict semi-stable  $S$ -curve and  $<$  be a total ordering on  $X^{(0)}$ . We denote by  $W = W(X, <, d)$  the model of  $(X_\eta)^d$  constructed in [algorithm 3.2](#). Then there is a morphism of graded rings defined by*

$$\varphi_W : \mathcal{C}(\Gamma(X)^d) \rightarrow \text{CH}_{W_s}^*(W), [C] \mapsto [C].$$

*Proof.* The relation (4.1) is trivial, relations (4.2) and (4.3) follow from [proposition 4.10](#) and [proposition 4.11](#).  $\square$

Let us focus on the main result of this section, a moving lemma on  $\mathcal{C}(\Gamma^d)$ . For this we use a definition of proper cycles in  $Z(\Gamma^d)$  similar to the one in intersection theory.

**Definition 4.15.** A monomial  $C_1 \cdots \cdots C_k \in Z(\Gamma^d)$  is called *proper*, if the vertices  $C_i \in \Gamma_0$  are pairwise different. An arbitrary element  $\alpha \in Z(\Gamma^d)$  is called *proper*, if it is a sum of proper monomials.

**Theorem 4.16.** *Let  $\Gamma$  be a connected finite graph without multiple simplices. Then the group  $\mathcal{C}^k(\Gamma^d)$  of the  $k$ -cycle classes is generated by the cycle classes of proper monomials*

$$\{C_1 \cdots \cdots C_k \mid C_1, \dots, C_k \in Z(\Gamma^d) \text{ pairwise different}\}.$$

Note that the special fibre  $X_s$  of a regular strict semi-stable  $S$ -curve is connected and so is the simplicial reduction set  $\mathcal{R}(X)$ .

Before we approach the proof, we define a useful decomposition of  $\Gamma^d$ :

*Remark 4.17.* Let  $\Gamma$  be a graph without multiple simplices. According to [proposition A.8](#) we identify the 1-simplices  $\gamma_1 \in \Gamma_1$  with morphisms  $i_{\gamma_1} : \Delta[1] \rightarrow \Gamma$ . Since  $\Gamma$  is without multiple simplices,  $i_{\gamma_1}$  is injective for each non-degenerate 1-simplex  $\gamma_1$ . Let now  $\gamma := (\gamma_1, \dots, \gamma_d) \in (\Gamma_1^{\text{nd}})^d$  be a  $d$ -tuple of 1-simplices. The product

$$i_\gamma := (i_{\gamma_1} \times \cdots \times \cdots \times i_{\gamma_d}) : I^d \rightarrow \Gamma^d$$

is injective as well and denoted by  $i_\gamma$ . The set of all  $i_\gamma$  gives a covering of  $\Gamma^d$ :

**Proposition 4.18.** *Let  $\Gamma$  be a finite connected graph without multiple simplices, which contains at least two vertices. Then the images of  $i_\gamma$  for  $\gamma \in (\Gamma_1^{\text{nd}})^d$  yield a covering of  $\Gamma^d$ , the covering by standard cubes. An arbitrary  $k$ -simplex  $\sigma \in (\Gamma^d)_k$  is in the image of  $i_\gamma$ , iff all vertices of  $\sigma$  are contained in the image of  $i_\gamma$ . If  $\sigma \in (\Gamma^d)_d^{\text{nd}}$  is a non-degenerate  $d$ -simplex, there is exactly one  $\gamma \in (\Gamma_1^{\text{nd}})^d$  such that  $\sigma$  is contained in the image of  $i_\gamma$ .*

*Proof.* elementary (see [\[Kol13\]](#))  $\square$

Let  $C_1 \cdots \cdots C_k$  be a monomial; for the proof of [theorem 4.16](#) we call the cardinality  $\#\{C_1, \dots, C_k\}$  the size of  $C_1 \cdots \cdots C_k$ . A monomial in  $\mathcal{C}^k(\Gamma^d)$  of size  $k$  is obviously a proper monomial. The proof is then carried out by induction on the maximum size of the monomials involved. We may reduce it to the following lemma:

**Lemma 4.19.** *Let  $\alpha = C_1 \cdots \cdots C_k \in Z^k(\Gamma^d)$  be a monomial of degree  $k$  and size  $l < k$ . Then there exists an element  $\alpha' \in Z^k(\Gamma^d)$  which consists of monomials of size  $l + 1$  or greater.*

The main ingredient for the proof is the following lemma, a kind of reduction to the standard cube:

**Lemma 4.20.** *Let  $\Gamma$  be a connected finite graph without multiple simplices,  $\gamma \in (\Gamma_1^{\text{nd}})^d$  and  $i_\gamma : I^d \rightarrow \Gamma^d$  the associated embedding according to [remark 4.17](#). We endow the vertices of the standard cube  $I^d$  with the product ordering and denote it by  $\leq$ . Let  $C_1, C_2 \in (I^d)_0$  be vertices of the standard cube with  $C_1 < C_2$ . Then the following holds:*

(i) *There is a 1-cycle  $\beta \in Z^1(\Gamma^d)$  such that*

$$i_\gamma(C_1)i_\gamma(C_2)^2 - \beta i_\gamma(C_1)i_\gamma(C_2) \in \text{Rat}(\Gamma^d)$$

holds and we have

$$i_\gamma^*(\beta) = \sum_{i=1}^n [E_i]$$

for a finite number of elements  $E_i \in I_0^d$  with  $E_i > C_1$  and  $E_i \neq C_2$ .

(ii) There is a 1-cycle  $\beta' \in Z^1(\Gamma^d)$  such that

$$i_\gamma(C_1)^2 i_\gamma(C_2) - \beta' i_\gamma(C_1) i_\gamma(C_2) \in \text{Rat}(\Gamma^d)$$

holds and we have

$$i_\gamma^*(\beta') = \sum_{i=1}^n [E'_i]$$

for a finite number of elements  $E'_i \in I_0^d$  with  $E'_i < C_2$  and  $E'_i \neq C_1$ .

*Proof.* Since  $C_1 < C_2$ , there exists  $j \in \{1, \dots, d\}$  such that  $\text{pr}_j(C_1) < \text{pr}_j(C_2)$ . We set

$$\tilde{\beta} := - \sum_{\substack{C' \in (\Gamma^d)_0 \setminus \{i_\gamma(C_2)\} \\ \text{pr}_j(C') = \text{pr}_j(i_\gamma(C_2))}} [C']$$

and get according to (4.3) in definition 4.12

$$(4.4) \quad i_\gamma(C_1) i_\gamma(C_2)^2 - \tilde{\beta} i_\gamma(C_1) i_\gamma(C_2) \in \text{Rat}(\Gamma^d).$$

Applying the injection  $i_\gamma$  we get

$$i_\gamma^*(\tilde{\beta}) = - \sum_{\substack{C \in (I^d)_0 \setminus \{C_2\} \\ \text{pr}_j(C') = \text{pr}_j(C_2)}} C.$$

Each of the elements  $C \in (I^d)_0 \setminus \{C_2\}$  with  $\text{pr}_j(C) = \text{pr}_j(C_2)$  is either bigger than  $C_1$  or not comparable with  $C_1$ . If  $C$  is not comparable with  $C_1$ , then  $C_1 C \in \text{Rat}(I^d)$  holds and according to proposition 4.18 also  $i_\gamma(C_1) i_\gamma(C) \in \text{Rat}(\Gamma^d)$ . Therefore we can remove these elements from  $\tilde{\beta}$  without changing the equation (4.4). The cycle

$$\beta := \sum_{\substack{C' \in (\Gamma^d)_0 \setminus \{i_\gamma(C_2)\} \\ \text{pr}_j(C') = \text{pr}_j(i_\gamma(C_2)) \\ C' \notin \{i_\gamma(C) \mid C \not\geq C_1\}}} C'$$

satisfies the claim.

The proof of (b) is done analogously.  $\square$

*Proof of lemma 4.19.* If  $\Gamma$  consists only of one vertex, the statement is trivial. In this case  $\Gamma^d$  has only one vertex and according to (4.2) in definition 4.12 each monomial of degree  $k \geq 2$  is in  $\text{Rat}(\Gamma^d)$ .

Let  $\alpha = \tilde{C}_1^{a_1} \dots \tilde{C}_l^{a_l} \in Z^k(\Gamma^d)$  be a monomial of degree  $k$  where  $C_1, \dots, C_l$  are different vertices of  $(\Gamma^d)_0$ . If  $l = 1$ , then a single application of (4.2) yields an equivalent cycle of size 2, i.e., consisting of monomials with at least two different factors. Thus we may assume from now on that  $l \geq 2$  and  $\Gamma$  is connected with at least two vertices.

If  $l \geq 2$  we can assume that  $\{\tilde{C}_1, \dots, \tilde{C}_l\}$  is a simplex in  $\Gamma^d$ , otherwise we would have  $\alpha = 0$ . According to remark 4.17 there exists an embedding of the standard cube  $i_\gamma : I^d \rightarrow \Gamma^d$  such that  $\tilde{C}_1, \dots, \tilde{C}_l$  are in its image. We denote the preimages with  $C_1, \dots, C_l$ . Since these preimages constitute a simplex in  $I^d$ , we may assume that  $C_1 < \dots < C_l$  holds (according to the product ordering in  $I^d$ ). We get

$$i_\gamma^* \alpha = C_1^{a_1} \dots C_l^{a_l}.$$

Let us denote by  $j(\alpha)$  the minimal index

$$j = j(\alpha) = \min\{j' \in \{1, \dots, l\} \mid a_{j'}(\alpha) \geq 2\}.$$

where a component occurs twice. It suffices to show that there is a 1-cycle

$$\beta = \sum_{\tilde{C}' \in (\Gamma^d)_0} b_{C'} C' \in Z^1(\Gamma^d), \quad b_{C'} \in \mathbb{Z}$$

such that

$$\alpha - \beta C_1^{a_1} \cdots C_j^{a_j-1} \cdots C_l^{a_l} \in \text{Rat}(\Gamma^d)$$

and  $b_{\tilde{C}_i} = 0$  holds for all  $i \in \{1, \dots, l\}$ . We show this once again by induction on  $j(\alpha)$ :

If  $j = j(\alpha) < l$ , we choose  $\beta'$  according to [lemma 4.20](#) (ii) such that

$$C_j^2 C_{j+1} - \beta' C_j C_{j+1} \in \text{Rat}(\Gamma^d)$$

with  $i^* \beta' = \sum E_i$  for  $C_j \neq E_i < C_{j+1}$ . We dissect the element

$$\beta C_1 \cdots C_j^{a_j-1} C_{j+1}^{a_{j+1}} \cdots C_l^{a_l}$$

into monomials  $\sum_i \alpha_i$ . Each monomial  $\alpha_i$  either contains an additional factor and has therefore a bigger size than  $\alpha$  or is of the form  $C_1 \cdots C_{j-1} C_j^{\tilde{a}_j} \cdots C_l^{\tilde{a}_l}$  with  $\tilde{a}_j < a_j$ . If we execute the same substitution with the latter monomials, we finally get  $\tilde{a}_j = 1$  and therefore  $j(\alpha_i) < j(\alpha)$ . The proposition follows then by induction.

On the other hand if  $j = j(\alpha) = l$ , we proceed analogous using [lemma 4.20](#) (i). We choose  $\beta' \in Z^1(\Gamma, <, d)$  such that

$$C_{j-1} C_j^2 - \beta' C_{j-1} C_j \in \text{Rat}(\Gamma^d)$$

with  $i^* \beta' = \sum_i E_i$  for  $C_j \neq E_i > C_{j-1}$  for all  $i$  holds. Then each monomial of the term

$$\beta C_1 \cdots C_j^{a_j-1} C_{j+1}^{a_{j+1}} \cdots C_l^{a_l}$$

has a bigger size than  $\alpha$ . □

**4.4. The Local Degree Map.** With the help of the moving lemma we can define a degree map on the combinatorial Chow ring, which is compatible with the degree map of [definition 4.6](#). We define it first for the standard 1-simplex  $I := \Delta[1]$ . This graph can be obtained as the reduction set of the proper regular strict semi-stable scheme  $\bar{L} := \text{Proj } R[z_0, z_1, t]/(z_0 z_1 - \pi t^2)$  (it is the projective completion of the standard scheme  $L$  considered in [section 3.2](#)). Denote by  $\bar{M}$  the desingularization of  $\bar{L}^d$  according to [algorithm 3.2](#). Since  $\bar{M}$  is a proper  $S$ -scheme, there exists a local degree map

$$\text{ldeg}_{\bar{M}} : \text{CH}_{\bar{M}_s}^{d+1}(\bar{M}) \rightarrow \mathbb{Z}$$

according to [definition 4.6](#). Furthermore we have  $\mathcal{R}(\bar{M}) = I^d$  and therefore a morphism of graded rings  $\varphi_{\bar{M}} : \mathcal{C}(I^d) \rightarrow \text{CH}_{\bar{M}_s}(\bar{M})$  by [proposition 4.14](#).

**Definition 4.21.** We call the morphism of  $\mathbb{Z}$ -modules

$$\text{ldeg}_{(I^d)} := \text{ldeg}_{\bar{M}} \circ \varphi_{\bar{M}} : \mathcal{C}(I^d) \rightarrow \mathbb{Z}$$

local degree map.

To define a local degree map for products of arbitrary graphs, we use the decomposition into standard cubes from [proposition 4.18](#):

**Definition 4.22.** Let  $\gamma$  be a finite graph and  $d \in \mathbb{N}$ . For each  $\gamma = (\gamma_1, \dots, \gamma_d) \in (\Gamma_1^{\text{nd}})^d$  denote by  $i_\gamma : I^d \rightarrow \Gamma^d$  the associated embedding of the standard cube as defined in [remark 4.17](#) and by  $i_\gamma^*$  the respective morphism of graded rings  $i_\gamma^* : \mathcal{C}(\Gamma^d) \rightarrow \mathcal{C}(I^d)$  as defined in [proposition 4.13](#). The *local degree map*  $\text{ldeg}_{\Gamma^d} : \mathcal{C}(\Gamma^d) \rightarrow \mathbb{Z}$  is then defined by

$$\text{ldeg}_{(\Gamma, <, d)} := \sum_{\gamma \in (\Gamma_1^{\text{nd}})^d} \text{ldeg}_{\bar{M}} \circ i_\gamma^*.$$

**Proposition 4.23.** Let  $X$  be a regular strict semi-stable curve over  $S$  with total ordering  $<$  on  $X^{(0)}$  and  $W := W(X, <, d)$  the associated model of the  $d$ -fold product given by [algorithm 3.2](#). We denote the morphism between the Chow ring and the combinatorial Chow ring by  $\varphi_W : \mathcal{C}(\Gamma(X)^d) \rightarrow \text{CH}_{W_s}(W)$ . Then the local degree map of  $\mathcal{C}(\Gamma(X)^d)$  coincides with the degree map of  $\text{CH}_{W_s}$ , i.e.,

$$\text{ldeg}_{(\Gamma(X)^d)} = \text{ldeg}_W \circ \varphi_W.$$

*Proof.* By [theorem 4.16](#) it suffices to show that the maps coincide for proper monomials. Let  $C_0, \dots, C_d \in \mathcal{R}(W)$  be different irreducible components of  $W_s$  with  $\{C_0, \dots, C_d\} \in \mathcal{R}_S(W)$ , thus  $C_0 \cap \dots \cap C_d \neq \emptyset$ . According to [proposition 4.8](#) all proper intersections in  $\text{CH}_{W_s}(W)$  have multiplicity 1, therefore  $\text{ldeg}_W \circ \varphi_W(C_0 \cdot \dots \cdot C_d) = \text{ldeg}_W([C_0] \cdot \dots \cdot [C_d]) = 1$  holds. To determine the left hand side, we examine the  $d$ -simplex given by the vertices  $(C_0, \dots, C_d)$ . Since this simplex is non-degenerate, there is exactly one tuple  $\gamma = (\gamma_1, \dots, \gamma_d) \in (\Gamma(X)^d)$  such that  $i_\gamma^* : I^d \rightarrow \Gamma^d$  maps the monomial  $C_0 \cdot \dots \cdot C_d$  to a proper monomial in  $\mathcal{C}(I^d)$  (see [proposition 4.18](#)). For each  $\gamma' \in (\Gamma_1^{\text{nd}})^d$  with  $\gamma \neq \gamma'$  one has  $i_{\gamma'}^*(C_0 \cdot \dots \cdot C_d) = 0$ . Therefore

$$\begin{aligned} \text{ldeg}_{(\mathcal{R}(X)^d)}(C_0 \cdot \dots \cdot C_d) &= \sum_{\bar{s} \in \Gamma_1^d} \text{ldeg}_{(I, <, d)} \circ i_{\bar{s}}^*(C_0 \cdot \dots \cdot C_d) \\ &= \text{ldeg}_{(I, <, d)} \circ i_s(C_0 \cdot \dots \cdot C_d) = 1. \end{aligned}$$

□

**4.5. Explicit Calculations in  $\mathcal{C}(I^d)$ .** By the preceding paragraph the calculation of  $\mathcal{C}(\Gamma^d)$  for arbitrary graphs can be reduced to the calculation of  $\mathcal{C}(I^d)$ , the product of the 1-simplex. We introduce first a convenient notation for the elements of the Chow ring and introduce an alternative generator set of  $\mathcal{C}(I^d)$  using a discrete fourier transform. For this alternative set of generators we deduce some relations.

We denote the vertices of  $I$  as usual with  $C_0$  and  $C_1$  and endow them with the ordering  $C_0 < C_1$ . For each vector  $v = (v_1, \dots, v_d) \in \mathbb{F}_2^d$  let  $C_v$  denote the vertex from  $I^d$  with  $\text{pr}_i(C_v) = C_{v_i}$ . Furthermore we call the vectors of the standard basis of  $\mathbb{F}_2^d$  as usual  $e_1 := (1, 0, \dots, 0), \dots, e_d := (0, \dots, 0, 1)$  and set  $E := \{e_1, \dots, e_d\}$ .

With this notation the set  $\{C_v \mid v \in \mathbb{F}_2^d\}$  is a generating set of  $\mathcal{C}(I^d)_{\mathbb{Q}} := (\mathcal{C}(I^d)) \otimes_{\mathbb{Z}} \mathbb{Q}$  and the relations from [definition 4.12](#) can be written in the following form:

$$(4.5) \quad C_v C_w = 0 \quad \begin{array}{l} \forall v, w \in \mathbb{F}_2^d : \exists i, j \in \{1, \dots, d\} \\ \text{with } (v_i, v_j) = (w_j, w_i) = (1, 0), \end{array}$$

$$(4.6) \quad \left( \sum_{w \in \mathbb{F}_2^d} C_w \right) C_v = 0 \quad \text{for all } v \in \mathbb{F}_2^d,$$

$$(4.7) \quad C_v C_{v'} \left( \sum_{\substack{w \in \mathbb{F}_2^d \\ w_i = v_i}} C_w \right) = 0 \quad \begin{array}{l} \text{for all } i \in \{1, \dots, d\} \text{ and } v, v' \in \mathbb{F}_2^d \\ \text{with } v_i \neq v'_i. \end{array}$$



*Remark 4.24.* For relation (4.5) it suffices to consider pairs  $\{v, w\}$  of vectors: If  $C_1, \dots, C_k \in (I^d)_0$  is a tuple with  $\{C_1, \dots, C_k\} \notin (I^d)_S$ , then there exists a pair  $i, j \in \{1, \dots, k\}$  such that  $\{C_i, C_j\} \notin (I^d)_S$ .

For the calculation of the local degree it suffices by [theorem 4.16](#) to consider proper intersections of  $d + 1$  elements, that is the product of  $d + 1$  pairwise different elements  $C_{v_0}, \dots, C_{v_d} \in \mathcal{C}(I^d)$ , such that  $\{C_{v_0}, \dots, C_{v_d}\} \in \mathcal{C}(I^d)_S$ . By the description of  $\mathcal{C}(I^d)$  in [corollary A.7](#) the last condition means that  $v_0, \dots, v_d$  is (up to permutation) an ascending chain according to the product ordering. Together with [proposition 4.8](#) the local degree is uniquely defined by setting

$$(4.8) \quad \text{ldeg}(C_{v_0} C_{v_1} \cdots C_{v_d}) = 1$$

if  $v_0 < v_1 < \dots < v_d$  according to the product ordering.

**Definition 4.25.** Let  $d \in \mathbb{N}$ . For each vector  $v \in \mathbb{F}_2^d$  we denote by  $F_v$  the element

$$F_v := \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v, w \rangle} C_w.$$

*Remark 4.26.* A straightforward computation in  $\mathcal{C}(I^d)_{\mathbb{Q}}$  shows

$$C_v = \frac{1}{2^d} \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v, w \rangle} F_w.$$

Therefore the set  $\{F_v \mid v \in \mathbb{F}_2^d\}$  is also a system of generators for  $\mathcal{C}(I^d)_{\mathbb{Q}}$ .

We deduce some useful relations of the  $F_v$ :

**Proposition 4.27.** Let  $v, v' \in \mathbb{F}_2^d$  and  $e, e' \in E$  be two base vectors of the standard basis. Then the following relations hold in  $\mathcal{C}(I^d)_{\mathbb{Q}}$ :

$$(4.9) \quad F_0 F_v = 0,$$

$$(4.10) \quad (F_{v+e+e'} - F_v)(F_{v'+e+e'} - F_{v'}) = (F_{v+e} - F_{v+e'})(F_{v'+e} - F_{v'+e'}),$$

$$(4.11) \quad F_e(F_v + F_{v+e})(F_{v'} - F_{v'+e}) = 0.$$

*Proof.* Equation (4.9) follows directly from (4.6) using  $F_0 = \sum_{v \in \mathbb{F}_2^d} C_v$ .

For the proof of (4.10) choose  $i, j \in \{1, \dots, d\}$  such that  $e = e_i, e' = e_j$  and denote by  $J \subseteq \mathbb{F}_2^d$  the subset  $J := \{w \in \mathbb{F}_2^d \mid w_i \neq w_j\}$ . Then one has

$$\begin{aligned} (F_v - F_{v+e+e'}) &= 2 \sum_{w \in J} (-1)^{\langle v, w \rangle} C_w, \\ (F_{v+e} - F_{v+e'}) &= 2 \sum_{w \in J} (-1)^{\langle v, w \rangle} (-1)^{\langle e, w \rangle} C_w. \end{aligned}$$

Therefore one calculates

$$\begin{aligned} & (F_v - F_{v+e+e'})(F_{v'} - F_{v'+e+e'}) - (F_{v+e} - F_{v+e'})(F_{v'+e} - F_{v'+e'}) \\ &= 4 \sum_{w, w' \in J} (-1)^{\langle v, w \rangle + \langle v', w' \rangle} (1 - (-1)^{\langle e, w \rangle + \langle e, w' \rangle}) C_w C_{w'} \\ &= 8 \sum_{\substack{w, w' \in J \\ w_i \neq w'_i}} (-1)^{\langle v, w \rangle + \langle v', w' \rangle} C_w C_{w'} = 0. \end{aligned}$$

The elements of the sum in the last equation vanish by (4.5) and the relations  $w_i \neq w'_i, w_j \neq w'_j$ .

After all, equation (4.11) can be deduced from

$$\begin{aligned}
 (4.12) \quad (F_0 + F_{e_i}) &= \sum_{\substack{w \in \mathbb{F}_2^d \\ w_i=0}} C_w, \\
 (F_v + F_{v+e_i}) &= \sum_{\substack{w \in \mathbb{F}_2^d \\ w_i=0}} (-1)^{\langle v, w \rangle} C_w \text{ and} \\
 (F_v - F_{v+e_i}) &= \sum_{\substack{w \in \mathbb{F}_2^d \\ w_i=1}} (-1)^{\langle v, w \rangle} C_w.
 \end{aligned}$$

By (4.7) one has

$$(F_v + F_{v+e_i})(F_w - F_{w+e_i})(F_0 + F_{e_i}) = 0$$

and by  $F_0 F_{v'} = 0$  for each  $v' \in \mathbb{F}_2^d$  one gets the claim.  $\square$

*Remark 4.28.* Since  $\{F_v \mid v \in \mathbb{F}_2^d\}$  is a system of generators for  $\mathcal{C}(I^d)_{\mathbb{Q}}$ , equation (4.9) already implies

$$F_0 \cdot \alpha = 0$$

for each element  $\alpha \in \mathcal{C}(I^d)_{\mathbb{Q}}$  with positive degree.

In the following situation the degree can be calculated generally:

**Proposition 4.29.** *In  $\mathcal{C}(I^d)_{\mathbb{Q}}$  the equation*

$$\text{ldeg}(F_v \prod_{i=1}^d F_{e_i}) = \begin{cases} (-4)^d & \text{if } v = (1, \dots, 1), \\ 0 & \text{otherwise} \end{cases}$$

*holds.*

*Proof.* For the proof we use the decomposition

$$F_v = \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v, w \rangle} C_w$$

and calculate for each vector  $w \in \mathbb{F}_2^d$  the value of  $\text{ldeg}(C_w \prod_{i=1}^d F_{e_i})$ .

For each  $t \in \{0, 1\}$  denote by  $\alpha^t(w)$  the product

$$\alpha^t(w) := \prod_{\substack{i \in \{1, \dots, d\} \\ w_i=t}} F_{e_i}.$$

Obviously we have  $\prod_{i=1}^d F_{e_i} = \alpha^0(w) \alpha^1(w)$ .

We will proof in a moment for each  $j$  with  $w_j \neq t$  the recursion

$$(4.13) \quad C_w \alpha^t(w) = 2(-1)^t C_w C_{w+e_j} \alpha^t(w + e_j).$$

Let us first show that this is enough to proof the claim: We may take a maximal increasing chain of vectors  $(0, \dots, 0) = w^{(0)} < \dots < w^{(d)} = (1, \dots, 1) \in \mathbb{F}_2^d$  which contains  $w$ . This means  $w^{(k)} = w$  for  $k = \langle w, (1, \dots, 1) \rangle$ . Multiple application of (4.13) shows

$$C_w \alpha^0(w) \alpha^1(w) = 2^d (-1)^{d-|w|} C_{w^{(0)}} \dots C_{w^{(d)}}.$$

We can therefore calculate the degree by (4.8)

$$\text{ldeg} \left( C_w \prod_{i=1}^d C_{e_i} \right) = 2^d (-1)^{d-|w|} = (-2)^d (-1)^{\langle w, (1, \dots, 1) \rangle}.$$

The claim now follows:

$$\begin{aligned} \text{ldeg} \left( F_v \prod_{i=1}^d F_{e_i} \right) &= \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v, w \rangle} \text{ldeg} \left( C_w \prod_{i=1}^d C_{e_i} \right) \\ &= \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v, w \rangle} (-2)^d (-1)^{\langle v, (1, \dots, 1) \rangle} \\ &= (-4)^d \delta_{v, (1, \dots, 1)}. \end{aligned}$$

It only remains to show (4.13). For this purpose we show that the difference

$$(4.14) \quad C_w \alpha^t(w) - 2(-1)^t C_w C_{w+e_j} \alpha^t(w + e_j)$$

vanishes. According to the precondition  $w_j \neq t$  we have  $\alpha^t(w) = F_{e_j} \alpha^t(w + e_j)$  and by remark 4.28

$$C_w F_{e_j} = (-1)^t C_w (F_0 + (-1)^t F_{e_j}) = (-1)^t C_w \left( 2 \sum_{\substack{w' \in \mathbb{F}_2^d \\ w'_j = t}} C_{w'} \right).$$

The second equation holds due to equation (4.12) in proposition 4.27. We conclude for the difference term (4.14)

$$\begin{aligned} &C_w \alpha^t(w) - 2(-1)^t C_w C_{w+e_j} \alpha^t(w + e_j) \\ &= C_w F_{e_j} \alpha^t(w + e_j) - 2(-1)^t C_w C_{w+e_j} \alpha^t(w + e_j) \\ &= 2(-1)^t C_w \alpha^t(w + e_j) \left( \sum_{\substack{w' \in \mathbb{F}_2^d \setminus \{w+e_j\} \\ w'_j = t}} C_{w'} \right) \end{aligned}$$

and therefore it suffices to show

$$C_{w'} C_w \alpha^t(w + e_j) = 0$$

for each  $w' \in \mathbb{F}_2^d$  with  $w'_j = t$  and  $w' \neq w + e_j$ . Since  $w' \neq w + e_j$ , there is another position  $k \in \{1, \dots, d\}$  besides  $j$ , in which  $w'$  and  $w$  differ. Assume  $t = 1$ : The conditions above imply  $w_j = 0, w'_j = 1$  and we can furthermore assume  $w_k = 0, w'_k = 1$ , since otherwise  $C_{w'} C_w = 0$  according to (4.5). Then (4.7) and (4.12) imply

$$C_{w'} C_w F_{e_k} = C_{w'} C_w (F_0 + F_{e_k}) = C_{w'} C_w \left( 2 \sum_{\substack{w'' \in \mathbb{F}_2^d \\ w''_k = 0}} C_{w''} \right) = 0.$$

The case  $t = 0$  is proven analogously. □

Another general proposition can be made using symmetries:

**Proposition 4.30.**

(i) *There is an isomorphism of graded rings,*

$$\psi : \mathcal{C}(I^d) \xrightarrow{\sim} \mathcal{C}(I^d),$$

*which is uniquely determined by  $\psi(C_v) = C_{v+(1, \dots, 1)}$ . The equation*

$$\psi(F_v) = (-1)^{\langle v, (1, \dots, 1) \rangle} F_v$$

*holds.*

- (ii) There is an operation of the symmetric group  $S_d$  onto  $\mathcal{C}(I^d)$ , which is for  $\sigma \in S_d$  uniquely determined by

$$\cdot^\sigma : \mathcal{C}(I^d) \rightarrow \mathcal{C}(I^d), C_v \mapsto C_{v^\sigma}.$$

The equation

$$(F_v)^\sigma = F_{v^\sigma}$$

holds.

Both automorphisms of graded rings are compatible with the local degree  $\text{ldeg}$ .

*Proof.* The existence and uniqueness of the morphisms  $\psi$  and  $\cdot^\sigma$  is evident by the definition of the combinatorial Chow ring written in the form of (4.5)-(4.7). The values of  $F_v$  can be calculated immediately. For the compatibility with  $\text{ldeg}$  it suffices to consider proper monomials (compare theorem 4.16). Let  $C_{v_0} C_{v_1} \cdots C_{v_d}$  be a non-trivial proper monomial. Without loss of generality we may assume that  $v_0, \dots, v_d$  form an ascending chain. Since the chains  $(v_0)^\sigma, \dots, (v_d)^\sigma$  and  $(v_d + (1, \dots, 1)), \dots, (v_0 + (1, \dots, 1))$  are then ascending as well, we can conclude by (4.8)

$$1 = \text{ldeg}_{\mathcal{C}(I^d)}(C_{v_0} \cdots C_{v_d}) = \text{ldeg}_{\mathcal{C}(I^d)}((C_{v_0} \cdots C_{v_d})^\sigma) = \text{ldeg}_{\mathcal{C}(I^d)}(\psi(C_{v_0} \cdots C_{v_d})).$$

□

**Corollary 4.31.** Let  $v_0, \dots, v_d \in \mathbb{F}_2^d$  with  $(\sum_{i=0}^d v_i) \cdot (1, \dots, 1) = 1$ . Then the equation

$$\text{ldeg}(F_{v_0} \cdots F_{v_d}) = 0$$

holds.

*Proof.* According to proposition 4.30 we have  $\text{ldeg} \circ \psi = \text{ldeg}$ . It follows

$$\begin{aligned} \text{ldeg}(F_{v_0} \cdots F_{v_d}) &= \text{ldeg}(\psi(F_{v_0} \cdots F_{v_d})) \\ &= (-1)^{\sum_{i=0}^d v_i \cdot (1, \dots, 1)} \text{ldeg}(F_{v_0} \cdots F_{v_d}) = -\text{ldeg}(F_{v_0} \cdots F_{v_d}) \end{aligned}$$

and therefore the claim. □

**4.6. Computation of  $\mathcal{C}(I^2)$  and  $\mathcal{C}(I^3)$ .** With the propositions of the last section we are able to compute all intersection numbers in the case  $d = 2$  and  $d = 3$ .

**Theorem 4.32.** Let  $d = 2$  and  $v_1, v_2, v_3$  vectors in  $\mathbb{F}_2^2$ . Then the following holds:

$$\text{ldeg}(F_{v_1} F_{v_2} F_{v_3}) = \begin{cases} -32 & \text{if } v_1 = v_2 = v_3 = (1, 1), \\ 16 & \text{if } \{v_1, v_2, v_3\} = \{(1, 0), (0, 1), (1, 1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* For brevity, we denote vectors of  $\mathbb{F}_2^2$  by concatenation of the digits, i.e.,  $10 = (1, 0)$ . The calculation  $\text{ldeg}(F_{10} F_{01} F_{11}) = (-4)^2 = 16$  was already done in proposition 4.29. By (4.9) and (4.11) we deduce from proposition 4.27 the equation

$$\begin{aligned} 2F_{10}^2 F_{11} &= F_{10}^2 ((F_{11} + F_{01}) + (F_{11} - F_{01})) \\ (4.15) \quad &= F_{10}(F_{10} - F_{00})(F_{11} + F_{01}) + F_{10}(F_{10} + F_{00})(F_{11} - F_{01}) \\ &= 0. \end{aligned}$$

Together with (4.10) we infer

$$F_{11}^3 = F_{11}(F_{11} - F_{00})^2 = F_{11}(F_{10} - F_{01})^2 = -2F_{10} F_{01} F_{11}$$

and thus  $\text{ldeg}(F_{11}^3) = -32$ . All other monomials vanish either by corollary 4.31 or equation (4.10) in proposition 4.27. □

The situation is more complicated in the case  $d = 3$ . We introduce some short-cuts in the notation: As in the previous proof we denote vectors of  $\mathbb{F}_2^3$  by concatenation of the digits such that 101 means the vector  $(1, 0, 1)$ . Additionally we order the vectors from  $\mathbb{F}_2^3$  first by the number of ones, then lexicographical:

$$000 \prec 100 \prec 010 \prec 001 \prec 110 \prec 101 \prec 011 \prec 111.$$

Four-tuples of vectors  $(v_0, v_1, v_2, v_3) \in (\mathbb{F}_2^3)^4$  are ordered lexicographical once again:  $(v_0, \dots, v_3) \prec (v'_0, \dots, v'_3)$  holds iff there is an  $N$  such that  $v_N \prec v'_N$  and  $v_i = v'_i$  for all  $i < N$ .

Let the symmetric group  $S_4$  operate on the four-tuples by permutation and the symmetric group  $S_3$  operate on the four-tuples by uniformly permuting each vector in the tuple: Set for each  $\sigma \in S_4, \tau \in S_3$

$$(v_0, \dots, v_3)^{\sigma, \tau} := (v_{\sigma(0)}^\tau, \dots, v_{\sigma(3)}^\tau).$$

According to [proposition 4.30](#) the equation

$$\text{ldeg}(F_{v_0} \cdots F_{v_3}) = \text{ldeg}(F_{v_{\sigma(0)}}^\tau \cdots F_{v_{\sigma(3)}}^\tau)$$

holds for all tuples  $(v_0, \dots, v_3) \in (\mathbb{F}_2^3)^4$ ,  $\sigma \in S_4$  and all  $\tau \in S_3$ . Hence it suffices to calculate the value of  $\text{ldeg}(F_{v_0} \cdots F_{v_3})$  only for the minimal element in the set

$$\{(v_0, v_1, v_2, v_3)^{\sigma, \tau} \mid \sigma \in S_4, \tau \in S_3\}.$$

**Theorem 4.33.** *Let  $V = (v_0, \dots, v_3) \in (\mathbb{F}_2^3)^4$  be a 4-tuple of vectors, such that*

$$(v_0, v_1, v_2, v_3) \leq (v_0, v_1, v_2, v_3)^{\sigma, \tau}$$

*holds for all  $\sigma \in S_4, \tau \in S_3$ . Then the intersection numbers in  $\mathcal{C}(I^3)_{\mathbb{Q}}$  are*

$$\text{ldeg}(F_{v_0} F_{v_1} F_{v_2} F_{v_3}) = \begin{cases} -64 & \text{if } V = (100, 010, 001, 111), \\ -64 & \text{if } V = (100, 010, 101, 011), \\ -64 & \text{if } V = (100, 110, 101, 111), \\ 128 & \text{if } V = (100, 011, 011, 111), \\ 128 & \text{if } V = (100, 111, 111, 111), \\ 128 & \text{if } V = (110, 110, 101, 011), \\ -128 & \text{if } V = (110, 101, 111, 111), \\ 512 & \text{if } V = (111, 111, 111, 111), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* For brevity set  $d(v_0, \dots, v_3) := \text{ldeg}(F_{v_0} \cdots F_{v_3})$  for any tuple  $(v_0, \dots, v_3) \in (\mathbb{F}_2^3)^4$ . First we deduce some equations: In analogy to [\(4.15\)](#) we get for each  $v \in \mathbb{F}_2^3$  and each unit vector  $e \in E$  the relation

$$\begin{aligned} (4.16) \quad 2F_e^2 F_v &= F_e^2 ((F_v - F_{v+e}) + (F_v + F_{v+e})) \\ &= F_e(F_e + F_0)(F_v - F_{v+e}) + F_e(F_e - F_0)(F_v + F_{v+e}) \\ &= 0. \end{aligned}$$

Together with [\(4.10\)](#) one deduces for each  $v \in \mathbb{F}_2^3$  the equation

$$(4.17) \quad F_{110}^2 F_v = (F_{110} - F_{000})^2 F_v = (F_{100} - F_{010})^2 F_v = -2F_{100} F_{010} F_v.$$

Furthermore by combining [corollary 4.31](#) and [\(4.10\)](#) we get the equation

$$(4.18) \quad d(e, u, v, w) = d(e, u + e, v + e, w)$$

for each  $u, v, w \in \mathbb{F}_2^3$  and  $e \in E$ .

We are now able to calculate the non-trivial values of  $d(v_0, \dots, v_3)$  in ascending order according to the order  $\prec$ .

**Tuples with  $\mathbf{v}_0 = 100, \mathbf{v}_1 = 010$ .** According to (4.9) and (4.16) the smallest tuple with non-trivial intersection number has to start with  $v_0 = 100, v_1 = 010, v_2 = 001$  and proposition 4.29 implies  $v_3 = 111$  and the value  $d(V) = (-4)^3 = -64$ . For the next possible tuple  $v_0 = 100, v_1 = 010, v_2 = 110$  (4.18) and (4.16) imply

$$d(100, 010, 110, v_3) = d(100, 010, 010, v_3 + 100) = 0 \quad \forall v_3 \in \mathbb{F}_2^3.$$

For the next in order,  $v_0 = 100, v_1 = 010, v_2 = 101$  we use once more (4.18) and get

$$d(100, 010, 101, v_3) = d(100, 010, 001, v_3 + 100).$$

By proposition 4.29 the only non-trivial result is  $v_3 = 011$ , thus

$$d(100, 010, 101, 011) = d(100, 010, 001, 111) = -64.$$

Tuples of the form  $v_0 = 100, v_1 = 010, v_2 = 101$  are not minimal under the operation of  $S_4$  and  $S_3$ .

It remains to examine  $d(100, 010, 111, 111)$ . Once again we use (4.18) and get

$$d(100, 010, 111, 111) = d(100, 010, 001, 001) = 0.$$

This gives a complete description of the case  $v_0 = 100, v_1 = 010$ .

**Tuples with  $\mathbf{v}_0 = 100, \mathbf{v}_1 \succ 010$ .** We can ignore tuples with  $v_1 = 001$ , since these are not minimal under the operations of  $S_4, S_3$ .

The next tuples in order have  $v_1 = 110$  and these can be reduced by

$$d(100, 110, v_2, v_3) = d(100, 010, v_2 + 100, v_3) = d(100, 010, v_2, v_3 + 100)$$

to combinations already calculated. Thus the only non-trivial combination of this type is  $V = (100, 110, 101, 111)$  with

$$d(100, 110, 101, 111) = d(100, 010, 001, 111) = -64.$$

Tuples with  $v_1 = 101$  are once again not minimal with respect to the  $(\sigma, \tau)$ -operation, so the next possible combination is  $V = (100, 011, v_2, v_3)$ . The minimality requires here  $011 \preceq v_2 \preceq v_3$ , thus  $v_2, v_3 \in \{011, 111\}$ . As corollary 4.31 yields  $d(100, 011, 111, 111) = 0$ , we may assume  $v_2 = 011$ . Then one deduces by (4.17)

$$d(100, 011, 011, v_3) = -2d(100, 010, 001, v_3)$$

and therefore a non-trivial result with  $v_3 = 111$  only.

The last possible tuple is  $V = (100, 111, 111, 111)$ . We deduce by (4.18):

$$d(100, 111, 111, 111) = d(100, 011, 011, 111) = 128.$$

So far we have covered all  $(\sigma, \tau)$ -minimal tuples starting with a vector from  $E = \{100, 010, 001\}$ .

**Remaining cases.** The next tuples in order are  $V = (110, 110, v_2, v_3)$ . These can be transformed by (4.17) into

$$d(110, 110, v_2, v_3) = -2d(100, 010, v_2, v_3)$$

and therefore yield a non-trivial result only with  $v_2 = 101, v_3 = 011$ :

$$d(110, 110, 101, 011) = -2d(100, 010, 101, 011) = 128.$$

If  $V$  is  $(\sigma, \tau)$ -minimal and contains the vector 110 only once, it also contains the vectors 101 and 011 only once. Therefore 111 is at least once in  $V$ . By corollary 4.31 we can only get a non-trivial result if 111 is contained in  $V$  twice or four times. Hence, the only remaining cases are  $V = (110, 101, 111, 111)$  and  $V = (111, 111, 111, 111)$ .

For the first one we deduce by (4.10)

$$\begin{aligned}
d(110, 101, 111, 111) &= \text{ldeg}(F_{110}F_{101}F_{111}F_{111}) \\
&= \text{ldeg}(F_{110}F_{101}(F_{111} - F_{001})^2) \\
&= \text{ldeg}(F_{110}F_{101}(F_{101} - F_{011})^2) \\
&= \text{ldeg}(F_{110}F_{101}F_{011}^2 - 2F_{110}F_{101}^2F_{011}) \\
&= -\text{ldeg}(F_{110}^2F_{101}F_{011}) \\
&= -128.
\end{aligned}$$

In this calculation, the already proven relations

$$\text{ldeg}(F_{110}F_{101}F_{001}^2) = \text{ldeg}(F_{110}F_{101}F_{001}F_{111}) = \text{ldeg}(F_{100}F_{110}F_{011}F_{111}) = 0$$

were used.

The case  $V = (111, 111, 111, 111)$  is shown by the analogous computation

$$\begin{aligned}
d(111, 111, 111, 111) &= \text{ldeg}(F_{111}^4) \\
&= \text{ldeg}(F_{111}^2(F_{111} - F_{100})^2 + 2F_{111}^3F_{100}) \\
&= \text{ldeg}(F_{111}^2(F_{110} - F_{101})^2 + 2F_{111}^3F_{100}) \\
&= -2\text{ldeg}(F_{110}F_{101}F_{111}^2) + 2\text{ldeg}(F_{100}F_{111}^3) \\
&= -2 \cdot (-128) + 2 \cdot 128 \\
&= 512.
\end{aligned}$$

□

**4.7. A Vanishing Condition.** The calculations in section 4.6 suggest that a lot of intersection products vanish. We make this observation precise in the following vanishing conjecture:

**Definition 4.34.** Let  $\mathcal{P} = \{P_1, \dots, P_l\}$  be a partition of the set  $\{1, \dots, d\}$  and  $v = (v_1, \dots, v_d) \in \mathbb{F}_2^d$ . Then set

$$\alpha(\mathcal{P}, v) := \#\{i \in \{1, \dots, l\} \mid \exists j \in P_i, v_j = 1\}.$$

**Definition 4.35** (vanishing condition). Let  $d \in \mathbb{N}$ . We say that  $d$  verifies the vanishing condition, iff for each partition  $\mathcal{P}$  of  $\{1, \dots, d\}$  and  $v_0, \dots, v_d \in \mathbb{F}_2^d$  with

$$\sum_i \alpha(\mathcal{P}, v_i) < d + |\mathcal{P}|$$

the intersection number

$$\text{ldeg}\left(\prod_i F_{v_i}\right)$$

vanishes.

Our calculations in section 4.6 verify the vanishing condition in two cases:

**Corollary 4.36.** For  $d = 2$  and  $d = 3$  the vanishing condition definition 4.35 is satisfied.

*Proof.* For the partition  $\mathcal{P} = \{\{1, \dots, d\}\}$  the condition definition 4.35 is always true, since  $\text{ldeg}(F_0 \cdot \prod_{i=1}^d F_{v_i}) = 0$  holds for all  $v_1, \dots, v_d \in \mathbb{F}_2^d$ .

If  $d = 2$ , there is only the partition  $\mathcal{P} = \{\{1\}, \{2\}\}$  left. For each  $v_1, \dots, v_3 \in \mathbb{F}_2^d$  with  $\text{ldeg}(F_{v_1}F_{v_2}F_{v_3}) \neq 0$  we have by theorem 4.32  $\sum_{i=1}^3 |v_i| \geq 4$ . Therefore definition 4.35 is true.



In the case  $d = 3$  it is easy to check in all non-trivial combinations of [theorem 4.33](#) that

$$\sum_i \alpha(\mathcal{P}, v_i) \geq d + |\mathcal{P}|$$

is satisfied.  $\square$

With the use of the computer algebra system Sage [\[Ste13\]](#) we are also able to verify the vanishing condition for  $d = 4$  and  $d = 5$ . Therefore we conjecture:

**Vanishing Conjecture.** *The vanishing condition is true for arbitrary  $d \in \mathbb{N}$ .*

#### APPENDIX A. THE CATEGORY OF SIMPLICIAL SETS

A good description for the simplicial structure of the special fibre in our setting is given by simplicial sets. For convenience we repeat the basic definitions and properties used in this paper.

**Definition A.1.** A partially ordered set is a set  $A$  endowed with a reflexive transitive and antisymmetric relation  $\leq$ . A morphism of partially ordered sets  $f : A \rightarrow B$  is a map of sets, which is monotonically increasing. This means  $a \leq a'$  implies  $f(a) \leq f(a')$  for each  $a, a' \in A$ .

*Remark A.2.* The category of partially ordered sets has finite products. The product of two partially ordered sets is given by the cartesian product  $A \times B$  endowed with the product order

$$(a, b) \leq (a', b') \iff a \leq a' \text{ and } b \leq b'$$

**Definition A.3.** Let  $\Delta$  denote the simplicial category. This has as objects the finite ordered sets  $[n] := \{0, \dots, n\}$  for each natural number  $n \in \mathbb{N}$  and monotonically increasing mappings as morphisms. A simplicial set  $\mathcal{R}$  is a contravariant functor  $\mathcal{R} : \Delta \rightarrow \text{Set}$ . For each  $n \in \mathbb{N}_0$  the set  $\mathcal{R}([n])$  is denoted by  $\mathcal{R}_n$  and its elements are called  $n$ -simplices. A morphism of simplicial sets is a strict transformation of functors. The category defined by this is called *category of simplicial sets* and denoted by  $\text{sSet}$ .

**Definition A.4.** The functor  $\Delta[n] := \text{Hom}_\Delta(\cdot, [n])$  is a simplicial set, the *standard  $n$ -simplex*.

**Definition A.5.** Let  $\mathcal{R}$  denote a simplicial set and  $k \in \mathbb{N}$ . A  $k$ -simplex  $\sigma \in \mathcal{R}_k$  is called degenerate if there exists a morphism  $d : [k] \rightarrow [k-1]$ , such that  $\sigma$  lies in the image of  $d^* : \mathcal{R}_{k-1} \rightarrow \mathcal{R}_k$ . The set of all nondegenerate simplices is denoted by  $\mathcal{R}_k^{\text{nd}}$ .

*Remark A.6.* By standard arguments of category theory (see [\[Awo10, Prop 8.7, Cor 8.9\]](#)) the category  $\text{sSet}$  has limits and colimits, which can be constructed component-by-component.

In particular one has the following description of products of standard-1-simplices:

**Corollary A.7.** *For each  $d \in \mathbb{N}$  there is a canonical bijection*

$$(\Delta[1])^d \simeq \text{Hom}_{\text{Poset}}(\cdot, [1]^d),$$

where  $[1]^d$  is seen as product of partially ordered sets.

*Proof.* Since the product  $(\Delta[1])^d$  can be constructed component-by-component we get for each  $n \in \mathbb{N}_0$  a functorial isomorphism

$$(\Delta[1])_n^d \simeq \prod_d \text{Hom}_\Delta([n], [1]) \simeq \text{Hom}_{\text{Poset}}([n], [1]^d)$$

as claimed.  $\square$

Furthermore each simplicial set is a colimit of standard simplicial sets:

**Proposition A.8.** *For each simplicial set  $\mathcal{R} \in \mathbf{sSet}$  the equations*

$$K_n \simeq \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n], K.)$$

and

$$K. = \mathrm{colim}_{\Delta K.} \Delta[n] = \mathrm{colim}_{\Delta' K.} \Delta[n]$$

hold.

*Proof.* This standard fact is proven for example in [Hov99, Lemma 3.1.3, Lemma 3.1.4].  $\square$

**Definition A.9.** For each  $i \in \{0, \dots, k\}$ , let  $s_i$  denote the morphism

$$s_i : [0] \rightarrow [k], 0 \mapsto i.$$

A simplicial set  $\mathcal{R}$  is called *simplicial set without multiple simplices*, if the map

$$\varphi : \prod_{k=0}^{\infty} K_k^{\mathrm{nd}} \rightarrow \mathcal{P}(K_0), t \in K_k^{\mathrm{nd}} \mapsto \{K(s_0)(t), \dots, K(s_k)(t)\}$$

is a monomorphism. If this is true, we denote the image of  $\varphi$  by

$$\mathcal{R}_S := \mathrm{Im}(\varphi) \subseteq \mathcal{P}(K_0).$$

We mostly consider only simplicial sets of this type, since the morphisms are uniquely defined by the mapping of the vertices:

**Proposition A.10.** *Let  $\mathcal{R}$  and  $\mathcal{R}'$  be two simplicial sets, where  $\mathcal{R}'$  has no multiple simplices and  $f, f' : \mathcal{R} \rightarrow \mathcal{R}'$  two morphisms of simplicial sets. If the restriction of  $f$  and  $f'$  onto the 0-simplices,*

$$f|_0, f'|_0 : K_0 \rightarrow K'_0,$$

*agree, then  $f = f'$  holds.*

*Proof.* It is enough to show that for each  $k \in \mathbb{N}$  and each nondegenerated  $k$ -simplex  $\sigma \in \mathcal{R}_k$  the equation  $f(\sigma) = f'(\sigma)$  holds. Since  $\mathcal{R}$  has no multiple simplices, the  $k$ -simplices  $f(\sigma)$  and  $f'(\sigma)$  are uniquely determined by  $\varphi(f(\sigma))$  and  $\varphi(f'(\sigma))$ . Since these elements depend only on  $f_0$  and  $f'_0$ , the proposition is true.  $\square$

For the description of ramified base-change we need a subdivision of simplicial sets. This can also be described completely categorially (see [Seg73, Appendix 1]):

**Definition A.11.**

- (i) Let  $k \in \mathbb{N}$ . We denote by  $\tilde{\mathrm{sd}}_k$  the functor

$$\tilde{\mathrm{sd}}_k : \Delta \rightarrow \Delta$$

given on objects by

$$[n] \mapsto [(n+1) \cdot k - 1]$$

and on morphisms by

$$\mathrm{Hom}_{\mathbf{sSet}}([n], [m]) \ni \varphi \mapsto (ak + b \mapsto ak + \varphi(b) \text{ for } 0 \leq b < k).$$

- (ii) The functor induced by  $\tilde{\mathrm{sd}}_k$

$$\mathrm{sd}_k : \mathbf{sSet} \rightarrow \mathbf{sSet}, K. \mapsto \tilde{\mathrm{sd}}_k \circ K.$$

is called the *k-fold subdivision functor*.

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